223 Cl[illegible] (Botha)

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Spiral Binding—U S Patents 1516[illegible]2-1942026. Other Patents Pending.

Physics 223 — Electric Waves. — Prof. Bethe.

Sept 26. References:

Vilbig — Hochfrequenztechnik. Good up to 1937

Since 1937: much development; esp. resonant cavities

Hansen — Jour. App. Phys.
: directed beams;

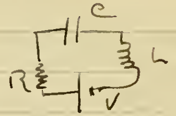
Course:

Alternating currents; quasi-stationary treatment;
Transmission lines;
Waves along conductors —
Emission from antennae.

Quasi-Stationary treatment.

Assume I, V function of time, + connected by circuit equation.

e.g. simple RLC circuit



$$V = RI + \frac{q}{C} + L\frac{dI}{dt}. \quad \text{(electromagnetic units)}$$

$$I = \frac{dq}{dt}.$$

Important case: periodic time dependence of V.
[if V is not sinusoidal, can be Fourier-analyzed anyhow, & each component treated separately].

Hence assume. $V = V_0 e^{i\omega t}$.

[Can introduce any phase desired by allowing V_0 to be complex].

Then $I = I_0 e^{i\omega t}$.

$$q = \frac{I_0}{i\omega} e^{i\omega t}.$$

Whence $$V_0 = I_0\left(R - \frac{i}{C\omega} + Li\omega\right)$$

$$\therefore \quad I_0 = \frac{V_0}{R + i\left(\omega L - \frac{1}{\omega C}\right)}.$$

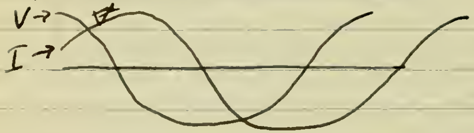
$$= \frac{V_0\left(R - i\left(\omega L - \frac{1}{\omega C}\right)\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

Or, let $R + i\left(\omega L - \frac{1}{\omega C}\right) = \mathcal{R}$. ("impedance")

(X = "reactance") $$\mathcal{R} = R + iX$$
$$= \sqrt{R^2 + X^2}\, e^{i\tan^{-1}\frac{X}{R}} = R_0 e^{i\varphi}$$

Then if V_0 is real, $$V = V_0 \cos\omega t$$
$$I = \left(R.P.\left(\frac{V_0}{R_0} e^{i(\omega t - \varphi)}\right)\right)$$
$$= \frac{V_0}{R_0}\cos(\omega t - \varphi).$$

φ positive:



Energy dissipated $$= \overline{VI} = \frac{V_0^2}{R_0}\,\overline{\cos\omega t\cos(\omega t - \varphi)}$$

$$= \frac{V_0^2}{R_0}\,\overline{\cos\omega t\cos\omega t\cos\varphi + \cos\omega t\sin\omega t\sin\varphi}$$

$$= \frac{1}{2}\frac{V_0^2}{R_0}\cos\varphi.$$

But $\frac{V_0}{R_0} = |I_0|$

$\therefore \overline{VI} = \frac{1}{2} |I_0|^2 R_0 \cos\varphi$

But $R_0 \cos\varphi$ = real part of R = R.

$\therefore \overline{VI} = \frac{1}{2} |I_0|^2 R$.

VI was not always positive; sometimes negative. This because besides Joule heat, there is electric energy in C, & magnetic energy in inductance; $W_{el} = \frac{q^2}{2C}$, $W_{mag} = \frac{1}{2} L I^2$. ; these increase & decrease periodically (building up & decay of el. & mag. fields) But Joule heat never comes back.

$$\frac{dW_{el}}{dt} = \frac{q}{C} I \quad ; \quad \frac{dW_{mag}}{dt} = L I \frac{dI}{dt}.$$

$$\frac{d}{dt}(W_{el} + W_{mag}) = I\left(\frac{q}{C} + L\frac{di}{dt}\right)$$

add Joule heat : $I \cdot IR$.

$$\text{Total} = I\left(\frac{q}{C} + L\frac{di}{dt} + IR\right) = IV.$$

Resonance : $\bar{X} = 0$.

or : $\omega L = \frac{1}{\omega C}$.

$$\omega = (LC)^{-1/2} .$$

Dimensions : $L \sim l$, $C \sim l^{-1} t^2$

$\therefore \omega \sim (l l^{-1} t^2)^{-1/2} = t^{-1}$.

Wave length emitted to free space : $\lambda = \frac{2\pi c}{\omega}$

$= 2\pi c \sqrt{LC}$.

But $C_{em} = \frac{C_{e.s.}}{c^2}$.

hence $\lambda = 2\pi \sqrt{L\, C_{e.s.}}$

$L \sim l$, $C_{e.s.} \sim l$, $\therefore \lambda$ obviously $\sim l$.

At Resonance R_0 is minimum ; $|I_0|$ maximum .

and $\varphi = 0$, $\therefore$ V + I are in phase .

If $R << \sqrt{\frac{L}{C}}$, very pronounced minimum in R_0 ,

" " maximum in $|I_0|$.

i.e. strong resonance .

Sept 29 | Want to investigate sharpness of minimum in R_0.

$$R_0^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 .$$

$$\omega_{res} L \frac{\omega}{\omega_{res}} - \frac{1}{C}\frac{\omega_{res}}{\omega\,\omega_{res}} = \sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)$$

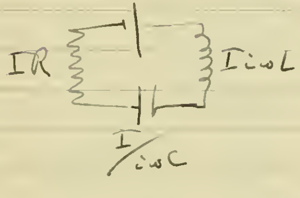
$$= \sqrt{\frac{L}{C}}\left(\frac{\omega^2 - \omega_{res}^2}{\omega\,\omega_{res}}\right).$$

If ω is of same order of mag. as ω_{res}, but different, then reactance $\sim \sqrt{\frac{L}{C}}$.

$\therefore$ Resonance is sharp if $R \ll \sqrt{\frac{L}{C}}$.

$\therefore$ define $Q = \frac{\sqrt{L/C}}{R}$; large Q means sharp resonance

Consider Physical meaning of $\sqrt{L/C}$:



EMF across L at resonant frequency $= I\omega_{res} L = I\sqrt{\frac{L}{C}}$.

EMF across C at resonant frequency $= \frac{I}{\omega_{res} C} = I\sqrt{\frac{L}{C}}$.

$\therefore \sqrt{\frac{L}{C}}$ is the apparent resistance of either L or C at the resonant frequency. If large Q, then large E.M.F. across both L & C at resonance, but they cancel each other —

Numerical example: Suppose we want a frequency $10^5 = 100$ kc.

then $\omega = 2\pi f = 6.3 \cdot 10^5$ sec^{-1}.

henry $= 10^9$ cm.
1 farad $= 9 \cdot 10^{11}$ cm.

$$LC = \frac{1}{\omega^2} = \frac{1}{4 \cdot 10^{6}} = .25 \cdot 10^{-6}$$

Suppose $C =$	1,	.10,	.01	μ farad.
then $L =$	2.5	25	250	μ henry.
and $Q = \sqrt{L/C} =$	1.6	16	160	ohms.

for L, need coil, which has R; for a coil of n turns, $L \sim n^2$, $R \sim n$; ∴ high L, low C is better, in general for high Q.

Free oscillations

Suppose V is turned off when $t = 0$, I known.

then subsequently: $q/C + RI + L\dot{I} = 0$.

assume $q = q_0 e^{\alpha t}$

then $\frac{1}{C} + R\alpha + L\alpha^2 = 0$

$$\alpha = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4L/C}}{2L}$$

$$= -\frac{R}{2L} \pm i\sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

Two solutions; 3 cases, $\sqrt{\ }$ real, 0 or imag.

a) $\frac{1}{CL} > \frac{R^2}{4L^2}$, or $R < 2\sqrt{\frac{L}{C}}$, or $Q > \frac{1}{2}$.

Then two solutions: $\alpha = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

damped oscillations. $= -\frac{R}{2L} \pm i\omega_0$

$$q \sim e^{\pm i\omega_0 t - \frac{Rt}{2L}}.$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \omega_{res}\sqrt{1 - \frac{1}{4Q^2}}.$$

$$\frac{R}{2L} = \omega_{res}\frac{R}{2\sqrt{L/C}} = \frac{\omega_{res}}{2Q}.$$

$$\therefore q \sim e^{\pm i\omega_0 t - \frac{\omega_{res} t}{2Q}}.$$

Damping for one period: $\frac{2\pi}{\omega_0} \cdot \frac{\omega_{res}}{2Q}$

$$= \frac{\pi}{Q}\frac{1}{\sqrt{1 - \frac{1}{4Q^2}}} = \frac{\pi}{\sqrt{Q^2 - 1/4}}$$

For large Q, damping $= \pi/Q$, for small damping, need high Q.

b) $\frac{R^2}{4L^2} > \frac{1}{LC}$. $R > 2\sqrt{\frac{L}{C}} \Rightarrow Q < \frac{1}{2}$.

Two roots: $\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$.

$$= -\frac{R}{2L}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

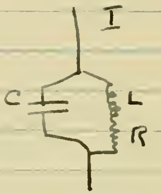
If $Q \ll 1$, $\alpha_1 = -\frac{R}{L}$, $\alpha_2\alpha_1 = \frac{1}{LC}$, hence $\alpha_2 = -\frac{1}{RC}$

$q = a_1 e^{-|\alpha_1| t} + a_2 e^{-|\alpha_2| t}$ at large t, $q \sim a e^{-|\alpha_2| t}$.

e.g. $C \sim 10^{-8}$ farad, $R = 1\,\Omega$, $|\alpha_1| = 10^{8}$ sec^{-1}
$L \sim 10^{-6}$ farad, $R = 1$ meg, $|\alpha_2| = 1$ sec^{-1}.

c) $\frac{1}{LC} = \frac{R^2}{4L^2}$, same root's, $q = e^{-\frac{R}{2L}t}(A_1 + A_2 t)$.

Oct. 1 Consider:



$$\frac{1}{\mathcal{R}} = \frac{1}{1/i\omega C} + \frac{1}{R + i\omega L}$$

$$\mathfrak{G} = i\omega C + \frac{1}{R + i\omega L}$$

$$= \frac{i\omega C R - \omega^2 LC + 1}{R + i\omega L}$$

Define $\mathfrak{G} = \frac{1}{\mathcal{R}}$; conductance.

$$\therefore \mathcal{R} = \frac{R + i\omega L}{1 - \omega^2 LC + iRC\omega}$$

$$\mathcal{R} = \frac{(R + i\omega L)(1 - \omega^2 LC - iRC\omega)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= \frac{R(1 - \omega^2 LC) + \omega^2 RLC + i\left(\omega L(1 - \omega^2 LC) - \omega R^2 C\right)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= \frac{R + i\omega\left(L - \omega^2 L^2 C - R^2 C\right)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

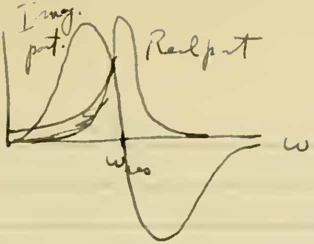
$\omega_{res} = \frac{1}{\sqrt{LC}}$

$Q = \frac{\sqrt{L/C}}{R}$

$$= \frac{R + i\omega L\left(1 - \frac{\omega^2}{\omega_{res}^2} - \frac{1}{Q^2}\right)}{\left(1 - \frac{\omega^2}{\omega_{res}^2}\right)^2 + \left(\frac{\omega}{Q\omega_{res}}\right)^2}$$

Real part of Impedance has sharp maximum at resonant frequency.

Imaginary part goes thru zero at resonant frequency —



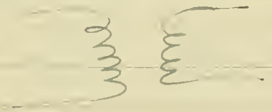
analogous to dispersion theory: R.P. to absorption,
I.P. to dielectric constant.

This circuit useful if a particular frequency is to be cut out of main circuit.

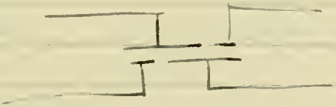
Coupled Circuits.

Can have: Inductive coupling
Capacitive "
Resistive "

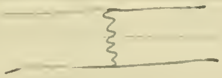
Inductive:



Capacitive:



Resistive



Inductive most common.

Mutual inductance M.

Equations of Inductively-coupled circuits:

$$L_1 \dot{I}_1 + R_1 I_1 + q_1 C_1^{-1} + M \dot{I}_2 = 0$$
$$L_2 \dot{I}_2 + R_2 I_2 + q_2 C_2^{-1} + M \dot{I}_1 = 0$$
(free oscillation)

Solutions of form: $I_1 = a_1 e^{\alpha t}$
$I_2 = a_2 e^{\alpha t}$.

$$\begin{cases} a_1\left(L_1\alpha + R_1 + \frac{1}{C_1\alpha}\right) + a_2 \alpha M = 0 \\ a_1 \alpha M + a_2\left(L_2\alpha + R_2 + \frac{1}{C_2\alpha}\right) = 0 \end{cases}$$

Sim. homog. eq.; hence sol'n only if Det $= 0$;

$$\therefore \left(L_1\alpha + R_1 + \frac{1}{C_1\alpha}\right)\left(L_2\alpha + R_2 + \frac{1}{C_2\alpha}\right) - \alpha^2 M^2 = 0.$$

Fourth order; in general four roots for α.

Special cases:

1) M small (loose coupling).
In particular, $M^2 << L_1 L_2$.

Then $L_{1(2)}\alpha + R_{1(2)} + \frac{1}{C_{1(2)}\alpha} \cong 0$

$$\therefore \alpha_{1(2)}' \cong \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.$$

" solution α_1', α_1''; α_2', α_2''.

Actual solutions are $\alpha = \alpha_1' + \beta_1'$ $\quad \beta_1' << \alpha_1'$.

then $\frac{1}{C_1\alpha^2} + R_1 + L_1\alpha_1 = \beta_1'\left(\frac{d}{d\alpha}(\ )\right) = \beta_1'\left(L_1 - \frac{1}{C_1\alpha_1'^2}\right)$

$$\therefore \quad \beta_1'\left(L_1 - \frac{1}{C_1\alpha_1'^2}\right) = \frac{M^2\alpha_1'^2}{\left(\frac{1}{C_2\alpha_1'} + R_2 + L_2\alpha_1'\right)}$$

Hence then solve for β_1'.

If R_1 is very small, $\alpha_1' = \pm i\,\omega_{res} = \pm \frac{i}{\sqrt{L_1C_1}}$.

$$\text{and} \quad L_1 - \frac{1}{C_1\alpha_1'^2} = L_1 - \frac{1}{C_1\left(-\frac{1}{L_1C_1}\right)} = 2L_1$$

$$\text{and} \quad \beta_1' = \frac{M^2\alpha_1'^2}{2L_1\left(L_2\alpha_1' + R_2 + \frac{1}{C_2\alpha_1'}\right)}.$$

Oct. 2 — Similar correction for each of the α's ($\alpha_1'', \alpha_2', \alpha_2''$)

Problem 1. Series C-L-R circuit.

L: 100 turns
1 cm. diameter
4 cm. length.
wire diameter = .1 cm, Cu.

} Calculate L & R. R of coil is only R.

Want res. frequency = 1 Mc = 10^6 cycle/sec. ($\omega = 2\pi f$)

What capacitance is necessary? What is Q?

Solution & from above; M zero, $\alpha_1', \alpha_1'', \alpha_2', \alpha_2''$;

$$L_1\alpha^2 + R_1\alpha + \frac{1}{C_1} = (\alpha - \alpha_1')(\alpha - \alpha_1'')L$$

same for α_2;

Hence Fundamental equation becomes:

$$(\alpha - \alpha_1')(\alpha - \alpha_1'')(\alpha - \alpha_2')(\alpha - \alpha_2'') = \frac{M^2}{L_1 L_2}\alpha^4 .$$

$\frac{M^2}{L_1 L_2}$ is called (coupling coefficient)^2; ideally $= 1$ when coils with large no of turns inside each other, so that field of one coil is concentrated inside it;

$$k^2 = \frac{M^2}{L_1 L_2}. \quad k = \text{coupling coeff.}$$

Weak coupling; k^2 small; hence at least one factor on left above must be small. or: Each "eigenvalue" must be close to an "unperturbed eigenvalue" α_1'' α_2', α_2'' α_2'.

Look for one close to α_1': $\alpha \cong \alpha_1'$

$$\text{Hence} \quad (\alpha - \alpha_1) = \frac{k^2 \alpha_1'^4}{(\alpha_1' - \alpha_1'')(\alpha_1' - \alpha_2')(\alpha_1' - \alpha_2'')}$$

O.K. if none of factors in denom. vanishes.

$\alpha_1' - \alpha_1''$ will in general not vanish, esp. in true oscillating circuit. (circuit 1) i.e. if circuit 1 has large Q; Then $\alpha_1' - \alpha_1'' = 2i\omega$; for large Q, $\alpha_1' = i\omega_0 - \beta_1$, $\beta_1 \approx \frac{\omega_0}{2Q}$, if large Q, $\beta_1 \ll \omega_0$; only if $Q = 1/2$ do we get double root.

$\alpha_1' - \alpha_2'$ might vanish; esp. if identical circuits, or if same frequency + damping; i.e if they are "tuned" to each other. Also possible

to have $\alpha_1'' \doteq \alpha_2''$ For Tuned circuit, formula breaks down.

For Tuned circuits, : $\alpha_1' \doteq \alpha_2'$, $\alpha_1'' \doteq \alpha_2''$;

Hence original eq. becomes:

$$(\alpha - \alpha_1')^2(\alpha - \alpha_1'')^2 = k^2\alpha^4 ;$$

$$\therefore \quad \alpha - \alpha_1 = \pm \frac{k\alpha^2}{\alpha - \alpha_1''} .$$

For Untuned circuit: $\Delta\alpha\,(\alpha - \alpha_1') \sim k^2$
" Tuned " : $\Delta\alpha \sim k$; also ambiguity of sign
i.e. new freq. may be *either* larger or smaller.
i.e. eigenvalue is *split* by the coupling.
Untuned circuits are *non-degenerate* (i.e. only current 1 responds to α_1) and perturbation is prop. to *square* of coupling; Tuned circuits are *degenerate*, & *perturbation* is ~~prop. to~~ linear in coupling.

Relative amplitudes : $I_1 = a_1 e^{\alpha t}$ $I_2 = a_2 e^{\alpha t}$;
then :

$$\begin{cases} (L_1\alpha^2 + R_1\alpha + \frac{1}{C_1})\,a_1 + M\alpha^2 a_2 = 0 \\ M\alpha^2 a_1 + (L_2\alpha^2 + R_2 + \frac{1}{C_2})\,a_2 = 0 \end{cases}$$

$$(\alpha - \alpha_1')(\alpha - \alpha_1'') + \frac{M}{L_1}\alpha^2 a_2 = 0$$

$$\downarrow$$

$$k\sqrt{\frac{L_2}{L_1}}\,\alpha^2 a_2$$

or
$$\begin{cases} (\alpha - \alpha_1)'(\alpha - \alpha_1'')\sqrt{L_1}\,a_1 + k\alpha^2\sqrt{L_2}\,a_2 = 0 \\ (\alpha - \alpha_2)'(\alpha - \alpha_2'')\sqrt{L_2}\,a_2 + k\alpha^2\sqrt{L_1}\,a_1 = 0 \end{cases}$$

~~If k is small,~~

Non-degenerate case (untuned circuits):

Take the α near α_1'; $(\alpha-\alpha_1') \sim k^2$

$\therefore$ Equation becomes:

$$a_1 \sqrt{L_1}\, k^2 + a_2 \sqrt{L_2}\, k \cong 0$$

$$\text{or } \frac{a_2 \sqrt{L_2}}{a_1 \sqrt{L_1}} \sim k .$$

$$\left(\text{Exactly}: \frac{k\, \alpha_1'^2}{(\alpha_1'-\alpha_2')(\alpha_1'-\alpha_2'')}.\right)$$

$\therefore$ $a_2 << a_1$. i.e. circuit I responds, since frequency is near its resonant freq; second circuit has current prop. to k - coupling coeff. Also, from exact expression, the more out of tune the circuits are, $(\alpha_1'-\alpha_2')$ large, very small response by second circuit.

If take $\alpha \sim \alpha_2'$, opposite happens.

Degenerate case (tuned circuits;)

Now $(\alpha-\alpha_1') \sim k$;

i.e. Both sides of expression are of order k, hence same order of amplitude!

$$(\alpha_1-\alpha_1') = \pm \frac{k\, \alpha_1'^2}{\alpha_1'-\alpha_1''}$$

$$\therefore \pm k\, \alpha_1'^2 \sqrt{L_1}\, a_1 + k\, \alpha_1'^2 \sqrt{L_2}\, a_2 = 0$$

$$\therefore \frac{a_1}{a_2} = \pm \sqrt{\frac{L_1}{L_2}} .$$

Amplitudes are equal if amplitude is defined as $(\alpha\sqrt{L})$ · ; either the same or opposite sign. Two "modes"; one frequency corr. to each one.

$\alpha_1' = i\omega, \quad \alpha_2'' = -i\omega$;

$$\text{then} \quad \alpha - i\omega = \pm \frac{k(i\omega)^2}{2i\omega}$$

$$= \pm\, i\tfrac{1}{2}k\,\omega .$$

$$\therefore \alpha = i\omega(1 \pm \tfrac{1}{2}k) .$$

higher freq. corr. to current in same direction;
lower " " " " " opposite " .

Oct. 6 '41

Case of Strong coupling: $k \approx 1$

[k can never be greater than 1; to see this, consider Magnetic Energy $\frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$; this is gotten from $U = \frac{1}{8\pi}\int H^2 d\tau$; since U is positive, Mag. Energy must be +, even tho I_1 + I_2 be opposite; Now write mag. energy as: $\frac{1}{2}(\sqrt{L_1}\, I_1)^2 + \frac{1}{2}(\sqrt{L_2}\, I_2)^2 \mp k(\sqrt{L_1}\, I_1)(\sqrt{L_2}\, I_2)$.
if $\sqrt{L_1}\, I_1 = \sqrt{L_2}\, I_2$, energy $= L_1 I_1^2(1 \mp k)$; hence $k \leq 1$.

Or: Mag. en. $= \frac{1}{2}\left[(\sqrt{L_1}\, I_1 - \sqrt{L_2}\, I_2)^2 + 2(1-k)\sqrt{L_1 L_2}\, I_1 I_2\right]$ + same result.]

Equation for characteristic frequencies was:

$$\left(L_1\alpha^2 + R_1\alpha + \frac{1}{C_1}\right)\left(L_2\alpha^2 + R_2\alpha + \frac{1}{C_2}\right) = M^2\alpha^4 .$$

Assume $R_1 \ll \sqrt{\frac{L_1}{C_1}}$, $R_2 \ll \sqrt{\frac{L_2}{C_2}}$; i.e. $Q_1 \gg 1$, $Q_2 \gg 1$

Hence neglect R_1, R_2 : $\left(L_1\alpha^2 + \frac{1}{C_1}\right)\left(L_2\alpha^2 + \frac{1}{C_2}\right) = M^2\alpha^4$;

or : $(L_1L_2 - M^2)\alpha^4 + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1}\right)\alpha^2 + \frac{1}{C_1C_2} = 0$

$k^2 = \frac{M^2}{L_1L_2}$ $= L_1L_2(1-k^2)\alpha^4 + \frac{L_1C_1 + L_2C_2}{C_1C_2}\alpha^2 + \frac{1}{C_1C_2} = 0$

or: $(1-k^2)\alpha^4 + \left(\frac{1}{L_2C_2} + \frac{1}{L_1C_1}\right)\alpha^2 + \frac{1}{L_1C_1L_2C_2} = 0$

$\omega_1^2 = \frac{1}{L_1C_1}$, $\omega_2^2 = \frac{1}{L_2C_2}$: $(1-k^2)\alpha^4 + (\omega_1^2 + \omega_2^2)\alpha^2 + \omega_1\omega_2 = 0$

Whence $\alpha^2 = \frac{-(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\omega_1\omega_2(1-k^2)}}{2(1-k^2)}$

since second term is less than first, both solutions are negative :

$\therefore$ write $\omega^2 = -\alpha^2 = \frac{\omega_1^2+\omega_2^2 \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + 4k^2\omega_1^2\omega_2^2}}{2(1-k^2)}$

[For small k, easily verified that same results as before obtain —].

For Tuned circuits : $\omega_1 = \omega_2$:

$$\omega^2 = \frac{2\omega_1^2 \pm 2k\omega_1^2}{2(1-k^2)} = \frac{\omega_1^2}{1 \mp k}$$

$$\text{or } \omega = \frac{\omega_1}{\sqrt{1 \mp k}} .$$

as $k \to 1$, the two frequencies approach $\frac{\omega_1}{\sqrt{2}}$ & ∞, resp.

Amplitudes:

$$a_1\left(L_1\alpha^2 + \frac{1}{C_1}\right) + M a_2 \alpha^2 = 0$$

$$a_1 \sqrt{L_1}\left(\alpha^2 + \frac{1}{L_1C_1}\right) + k\sqrt{L_1L_2}\,\alpha^2 a_2 = 0$$

$$a_1\sqrt{L_1}\left(-\omega^2 + \omega_1^2\right) - k\sqrt{L_2}\,a_2\,\omega^2 = 0$$

$$\therefore \quad \frac{a_2\sqrt{L_2}}{a_1\sqrt{L_1}} = \frac{\omega_1^2 - \omega^2}{k\omega^2}$$

Tuned circuits: $\omega^2 = \dfrac{\omega_1^2}{1 \pm k}$:

$$\therefore \quad \frac{a_2\sqrt{L_2}}{a_1\sqrt{L_1}} = \frac{\omega^2(1\pm k) - \omega^2}{k\omega^2}$$

$$= \pm 1.$$

Hence equal amplitude; in phase for low frequencies, out of phase for high freq.

Criterion for tuning: from equation for ω^2:

Circuits tuned if: $(\omega_1^2 - \omega_2^2) < 2k\,\omega_1\omega_2$;

Or, if k is not too large, $(\omega_1^2 - \omega_2^2) < k(\omega_1+\omega_2)^2 \cdot \frac{1}{2}$

or $\dfrac{\omega_1 - \omega_2}{\frac{1}{2}(\omega_1+\omega_2)} < k$.

i.e. difference in freq. divided by average freq. $< k$.

Let ω', ω'' be frequencies of coupled circuits;

then $\omega'^2 - \omega''^2 = \dfrac{\sqrt{(\omega_1^2-\omega_2^2)^2 + 4k^2\omega_1^2\omega_2^2}}{(1-k^2)} > |\omega_1^2 - \omega_2^2|$

i.e. difference in frequencies of system coupled,

is greater than difference in natural frequencies;
(Just as in Q-M., a perturbation always pushes almost degenerate levels further apart).

Problem 2. Find damping in case of strong coupling.
By: Treat resistance as a perturbation; say actual solution α lies close to sol'n's without resistance.
(Use Q's).

Forced oscillations of coupled circuits.

Assume Circuit 1 has E.M.F. (sinusoidal).

then:

$$\begin{cases} L_1 \dot{I}_1 + R_1 I_1 + \frac{q_1}{C_1} + M \dot{I}_2 = E = E_0 e^{i\omega t} \\ L_2 \dot{I}_2 + R_2 I_2 + \frac{q_2}{C_2} + M \dot{I}_1 = 0 \end{cases}$$

Put $I_1 = a_1 e^{i\omega t}$, $I_2 = a_2 e^{i\omega t}$

then:

$$\begin{cases} a_1\left(i\omega L_1 + R_1 - \frac{i}{\omega C_1}\right) + a_2 M i\omega = E_0 \\ a_2\left(i\omega L_2 + R_2 - \frac{i}{\omega C_2}\right) + a_1 M i\omega = 0 \end{cases}$$

or:

$$\begin{cases} a_1\left(-\omega^2 L_1 + i\omega R_1 + \frac{1}{C_1}\right) - a_2 \omega^2 M = i\omega E_0 \\ -a_1 \omega^2 M + a_2\left(-\omega^2 L_2 + i\omega R_2 + \frac{1}{C_2}\right) = 0. \end{cases}$$

solve for a_1: $a_1\left\{\left(-L_1\omega^2 + i\omega R_1 - \frac{1}{C_1}\right)\left(-L_2\omega^2 + iR_2\omega + \frac{1}{C_2}\right) - M^2\omega^4\right\}$

$$= i E_0 \omega\left(-L_2\omega^2 + iR_2\omega + \frac{1}{C_2}\right)$$

{ } : characteristic quantity for free osc.

( ) : " " " " - of 2nd

Let $\omega_1\ \omega_2\ \omega_3\ \omega_4$ be char. freq. of coupled circuit.

" ω_5, ω_6 " " " circuit 2:

then: $(L_1 L_2 - M^2)(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)\, a_1$
$$= -i E_0 \omega L_2 (\omega-\omega_5)(\omega-\omega_6)$$

∴ Resonance (a_1 max.) if $\omega \simeq$ one of char. freq. of coupled circt; but a_1 vanishes if $\omega \approx$ one of freq. of 2nd circuit!

For small R_1, R_2, $\omega_3 \sim -\omega_1$, $\omega_4 \sim -\omega_2$;
$\omega_6 \sim \omega_5$

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$\omega_5 = \omega_5' + i\beta_5$ $\qquad$ $\omega_1 = \omega_1' + i\beta_1$ $\qquad$ $\omega_2 = \omega_2' + i\beta_2$
$\omega_6 = -\omega_5' + i\beta_5$ $\qquad$ $\omega_3 = -\omega_1' + i\beta$ $\qquad$ $\omega_4 = -\omega_2' + i\beta_2$

$$\text{Then } a_1 = \frac{-iE_0\omega L_2}{L_1L_2 - M^2} \cdot \frac{(\omega-\omega_5'-i\beta_5)(\omega+\omega_5'-i\beta_5)}{(\omega-\omega_1'-i\beta_1)(\omega+\omega_1'-i\beta_1)(\omega-\omega_2'-i\beta_2)(\omega+\omega_2'-i\beta_2)}$$

Resonances: ω_1', ω_2' large current
ω_5' small current (large e.m.f.)

(This combines features of series + parallel resonance)
$\qquad\qquad\qquad\qquad\quad \downarrow \qquad\qquad \downarrow$
$\qquad\qquad\qquad\qquad \omega_1' \; \omega_2' \qquad\quad \omega_5'$

Here we have:

① can have series resonance at its own at ω_1' & ω_2';
② has EMF impressed on it from outside (like a ‖ series; so ‖ resonance at ω_5'.

$$\frac{a_2}{a_1} = + \frac{iM\omega^2}{(-\omega^2 L_2 + iR_2\omega + \frac{1}{C_2})} - \frac{+\xi M\omega^2}{(\omega-\omega_5)(\omega-\omega_6)}$$

$$\therefore\ a_2 \sim \frac{M\omega^2}{(\omega-\omega_1)(\omega-\omega_2)(\ --\ )(\ \ )}$$

i.e. series resonance for ω_1', ω_2'.

More complicated circuits get too complicated; sometimes do them graphically — cf. Vilbig —.

<u>Transformer</u>. example of coupled circuit.

Usually use smaller frequencies than resonance; don't use resonance properties.

For $\omega \gg \omega_5$, $\frac{a_2}{a_1} = -\frac{M}{L}$.

" $\omega \ll \omega_5$ $\frac{a_2}{a_1} \simeq MC_2\omega^2$.

If $\omega \ll \omega_1', \omega_2' \omega_5'$, $a_1 \cdot \frac{1}{C_1C_2} = \frac{i\omega E_0}{C_2}$

$\therefore a_1 \simeq i\omega E_0 C_1$.

$a_2 \simeq i\omega^3 E_0 C_1 C_2 M$.

Potential diff in second circuit: $\frac{a_2}{iC_2\omega} = \omega^2 E_0 C_1 M$.

For $\frac{1}{C_1}, \frac{1}{C_2}$ small, resonant frequencies very small;

then $\omega \gg \omega_1' \omega_2' \omega_5'$ (the usual case)

and $a_1 = -\frac{iE_0}{\omega} \frac{L_2}{L_1L_2 - M^2} = \frac{E_0}{iL_1\omega} \frac{1}{(1-k^2)}$

$a_2 = -\frac{iE_0}{\omega} \frac{M}{L_1L_2 - M^2} = -\frac{M}{L_2} \frac{E_0}{\omega L_1} \frac{1}{1-k^2}$.

For strong coupling, $M^2 \sim L_1L_2$

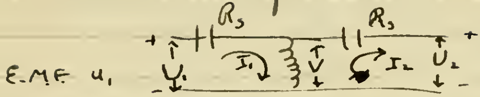
and $-\frac{a_2}{a_1} = \frac{M}{L_2} \sim \sqrt{\frac{L_1}{L_2}}$.

~~Pot. diff : $i\omega L_2 a_2 = \frac{E_0 L_2 M}{L_1L_2 - M^2}$~~

Since $L \sim n^2$, $\frac{a_2}{a_1} \sim \sqrt{\frac{L_1}{L_2}} \sim \frac{n_1}{n_2}$.

Filters. circuits which will not (or will) transmit certain frequencies.

Can make filters so:



R_c = impedance of coil;

$$V = (I_1 - I_2) R_c$$
$$U_1 = V + I_1 R_s$$
$$U_2 = V - I_2 R_s$$

Want I_2, u_2 in terms of I_1, U_1.

$$-I_2 = \frac{V}{R_c} - I_1$$
$$= \frac{U_1 - I_1 R_s}{R_c} - I_1$$
$$I_2 = -\frac{U_1}{R_c} + I_1\left(\frac{R_s}{R_c} + 1\right)$$
$$U_2 = (U_1 - I_1 R_s) + \left[\frac{U_1}{R_c} - I_1\left(1 + \frac{R_s}{R_c}\right)\right] R_s$$
$$= u_1\left[1 + \frac{R_s}{R_c}\right] - I_1\left[R_s + R_s\left(1 + \frac{R_s}{R_c}\right)\right]$$
$$= u_1\left[1 + \frac{R_s}{R_c}\right] - I_1 R_s\left[2 + \frac{R_s}{R_c}\right].$$

Oct. 10, '41 (change signs above)

Want solutions where ratio of current to potential is the same at each end, so as to simplify calculation of effect due to whole series of filters.

$$\text{Want} \quad \frac{u_2}{I_2} = \frac{u_1}{I_1}. \quad ; \text{ call these characteristic sol'ns.}$$

$$\text{then} \quad \frac{u_2}{u_1} = \frac{I_2}{I_1} = \lambda.$$

Put in eq'n:

$$\begin{cases} I_2 = -\frac{u_1}{R_c} + I_1\left(\frac{R_s}{R_c} + 1\right) = \lambda I_1 \\ u_2 = u_1\left(1 + \frac{R_s}{R_c}\right) - I_1 R_s\left(2 + \frac{R_s}{R_c}\right) = \lambda u_1 \end{cases}$$

or:

$$\begin{cases} \left(1 + \frac{R_s}{R_c} - \lambda\right) I_1 = \frac{u_1}{R_c} \\ \left(1 + \frac{R_s}{R_c} - \lambda\right) u_1 = I_1 R_s\left(2 + \frac{R_s}{R_c}\right) \end{cases}$$

For solutions, Det. of coeff. must vanish:

$$\left(1+\frac{R_s}{R_c}-\lambda\right)^2 = \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right).$$

$$\therefore \lambda = 1+\frac{R_s}{R_c} \pm \sqrt{\frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right)}.$$

Product $\lambda_1\lambda_2 = \left(1+\frac{R_s}{R_c}\right)^2 - \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right) = 1.$

Hence one λ gives increase in amplitude.
other λ " decrease " "

Set $\cosh^2 g = \left(1+\frac{R_s}{R_c}\right)^2$

$\sinh^2 g = \frac{R_s}{R_c}\left(2+\frac{R_s}{R_c}\right).$

then $\lambda = \cosh g \pm \sinh g.$

$$\therefore \begin{cases}\lambda_1 = e^{g} \\ \lambda_2 = e^{-g}\end{cases}.$$

Now $\cosh g = 1+\frac{R_s}{R_c} = 1+2\sinh^2 \frac{g}{2}.$

$$\therefore \underline{\sinh\frac{g}{2} = \sqrt{\frac{R_s}{2R_c}}}$$

Now find $\frac{u_2}{I_2} = \frac{u_1}{I_1}$:

$$\frac{u_1}{I_1} = \mp Z = R_c\left(1+\frac{R_s}{R_c}-\lambda\right)$$

$$= R_c(\cosh g - \lambda).$$

Z called wave resistance

$Z = R_c \sinh g.$

$$= R_c\left(\cosh g - e^{\pm g}\right)$$

$$= \mp R_c \sinh g$$

Suppose we have a long line of filters; at end of
then $\mathfrak{U}_n$ = pot. in n^{th} element
I_n = current " " "

And $\mathfrak{U}_n = -aZe^{ng} + bZe^{-ng}$
$I_n = ae^{ng} + be^{-ng}$

Suppose line ends with working impedance R_e;

then $\mathfrak{U}_n = R_e I_n$;

or: $-aZe^{ng} + bZe^{-ng} = aR_e e^{ng} + bR_e e^{-ng}$

$a(-Z - R_e)e^{ng} = b(R_e - Z)e^{-ng}$

or $\frac{a}{b} = \frac{Z - R_e}{Z + R_e} e^{-2ng}$

if large n, R.P. $g > 0$, and $Z \not\approx -R_e$,
then $\frac{a}{b} \ll 0$

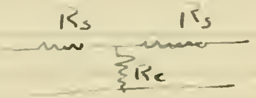
i.e. a <u>decrease</u> in current & EMF.

this determines b: $\mathfrak{U}_0 = bZ$
$\therefore b = \frac{\mathfrak{U}_0}{Z}$.

For $Z + R_e \cong 0$: get resonance;
get sizable a, hence possibly amplification.

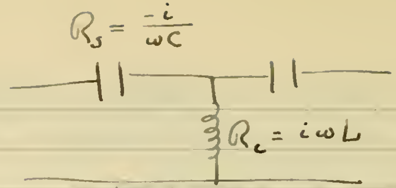
Some Possible filters:

1): Pure resistances



then $\sinh g/2 = \sqrt{\frac{R_s}{2R_e}}$ and g is purely real;
and $\neq 0$.

Same damping for any frequency; hence no good for wave filter.

2) 

$$\sinh q/2 = \sqrt{\frac{R_s}{2R_c}} = \sqrt{\frac{-i/\omega C}{2i\omega L}} = \sqrt{\frac{1}{-2\omega^2 LC}} = -i\sqrt{\frac{1}{2\omega^2 LC}}.$$

Let $\frac{1}{2CL} = \omega_1^2$

then $\sinh q/2 = -i\,\frac{\omega_1}{\omega}$.

$q/2 = b + ia$

$\sinh(b+ia) = -i\,\frac{\omega_1}{\omega}$

$\sinh b \cos a + i \cosh b \sin a = -i\,\frac{\omega_1}{\omega}$

$\therefore \sinh b \cos a = 0$

$\therefore \sinh b = 0$ or $\cos a = 0$

$b = 0$ or $a = \pm \pi/2$.

also $\cosh b \sin a = -\frac{\omega_1}{\omega}$.

if $\frac{\omega_1}{\omega} < 1$, must take $b = 0$, $\cosh b = 1$

$\therefore \sin a = -\frac{\omega_1}{\omega}$

if $\frac{\omega_1}{\omega} > 1$ must take $a = \pm\pi/2$, $\sin a = \pm 1$

$\therefore \cosh b = \frac{\omega_1}{\omega}$

i.e. $\therefore$ $\omega_1 > \omega$: $a = -\pi/2$
$b = \cosh^{-1}\frac{\omega_1}{\omega}$, $b > 0$ { have get damping

$\omega_1 < \omega$: $b = 0$
$a = \sin^{-1}\left(-\frac{\omega_1}{\omega}\right)$, $0 < a < -\pi/2$. { no damping.

$(q = 2b + i2a)$. Transmits short waves }
absorbs long waves }.

Oct. 13, '41 $\quad I_2 = I_1 e^{-g}$ (if g has a positive real part)
(we showed other part $I_1' e^{+g}$ small for large g)

If real part of g vanishes, then the solutions $I_1 e^{-g}$ & $I_1' e^{+g}$ are equally likely — $\therefore \; I_2 = I_1 e^{-g} + I_1' e^{+g}$

Hence for $\omega > \omega_1$, we have $\quad I_2 = I_1 e^{-2ia}$

$$I_2' = I_1' e^{+2ia}$$

Hence only a phase shift of $2a$.

For: $\quad \omega = \infty \quad , \quad \omega_1$

$\quad\quad a = 0 \quad , \quad \pi/2$

Phase shift $= 0 \quad , \quad \pi$ ($\rightarrow$ i.e. current opposite in successive elements)

Ampl. ratio: $\quad 1 \quad , \quad 1$

For $\omega < \omega_1$, $\quad a = -\pi/2$, phase shift $= -\pi$.

~~For ω~~ $\quad \cosh b = \frac{\omega_1}{\omega} = \frac{1}{2}(e^{b} + e^{-b})$

Let $x = e^{b}$:

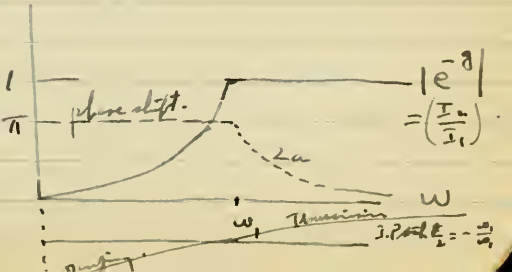
$$x \frac{\omega_1}{\omega} = \frac{1}{2}(x^2 + 1)$$

Hence for low frequencies, amplitude ratio in successive elements is $\left(\frac{2\omega_1}{\omega}\right)^2$; very strong damping.

$$\therefore \; x = \frac{\omega_1}{\omega} \pm \sqrt{\left(\frac{\omega_1}{\omega}\right)^2 - 1}$$

$$\therefore \; e^{g} = -e^{2b} \approx \left(\frac{2\omega_1}{\omega}\right)^2$$

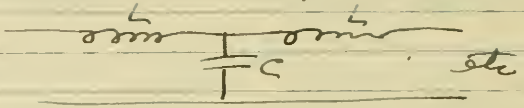
$$\omega_1 \gg \omega.$$



That filter not too good; only damps long waves — very long ones at that.

Suppose we want to damp short waves:

Take:



then $R_s = i\omega L$

$R_c = -\frac{i}{\omega C}$

$\sinh \gamma/2 = \sqrt{\frac{\;}{4i^2\omega^2 LC}} = i\,\frac{\omega}{\omega_1}$.

$\omega_1 = \sqrt{\frac{1}{LC}}$.

Just the reverse of previous case;

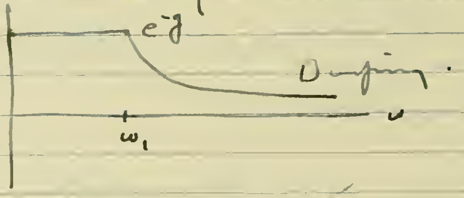
$\therefore\ \omega > \omega_1$ gives damping

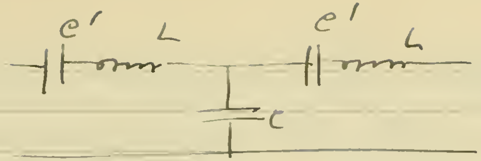
$\omega < \omega_1$ gives transmission.

For $\omega < \omega_1$: $\sin a = \omega/\omega_1$; $b = 0$

For $\omega > \omega_1$: $\cosh b = \omega/\omega_1$; $a = \pi/2$ (phase lift π)

$$e^{\gamma} = \left[\frac{\omega}{\omega_1} + \sqrt{\left(\frac{\omega}{\omega_1}\right)^2 - 1}\;\right]^2 \approx \left(\frac{2\omega}{\omega_1}\right)^2 \text{ for large } \omega \gg \omega_1$$





then $R_s = i\omega L + \frac{1}{i\omega C'}$

$R_c = \frac{1}{i\omega C}$

then $\sinh \gamma/2 = \sqrt{\frac{R_s}{2R_c}} = \sqrt{\frac{1}{2}i\left(\omega L - \frac{1}{\omega C'}\right) i\omega C}$

$= \pm \sqrt{\frac{\omega^2 LC}{2} + \frac{C}{2C'}}$

For small ω, $\frac{R_s}{R_c} \approx \frac{C}{C'}$

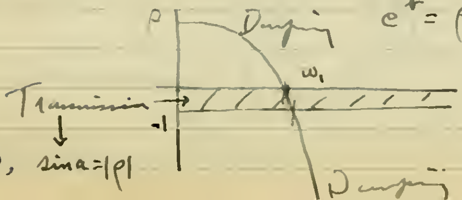
then $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}}$ can take $C \gg C'$;

where $\omega = 0$, $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}}$ ∴ large damping constant while $\frac{C}{C'} \gg 1$

write $\frac{1}{C'L} = \omega_1^2$

then $\sinh \gamma/2 = \sqrt{\frac{C}{2C'}\left(1 - \frac{\omega^2}{\omega_1^2}\right)}$

hence as $\omega \to \omega_1$, $\sinh \gamma/2 \to 0$; come into region of transmission; then after $\sinh \gamma/2$ becomes $< \sqrt{-1}$, get damping again.



$e^t = \rho + \sqrt{\rho^2+1}$, $a = 0$

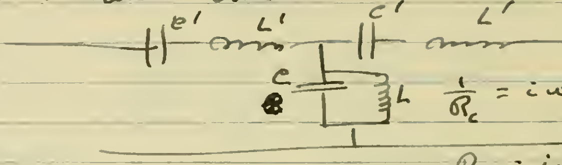
$\frac{R_s}{2R_c} = \rho$

$e^t = |\rho| + \sqrt{\rho^2-1}$, $2a = \pi$.

this has drawback that damping is constant at low ω.

Would like a filter that transmits a narrow band like this one, but damps low frequencies more effectively than this one.

Can be done thus:



$$\frac{1}{R_c} = i\omega C + \frac{1}{i\omega L}$$

$$R_s = i\omega L' + \frac{1}{i\omega C'}$$

this will give → ∞ damping for both low + high freq.

$$\rho = \frac{R_s}{2R_c} = \frac{1}{2}\left(i\omega C + \frac{1}{i\omega L}\right)\left(i\omega L' + \frac{1}{i\omega C'}\right)$$

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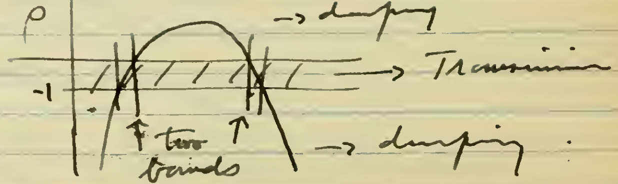
$$= \frac{1}{2}\left(-\omega^2 C L' - \frac{1}{\omega^2 L C'} + \frac{C}{C'} + \frac{L'}{L}\right)$$

∴ $\rho \to -\infty$ for $\omega \to$ either ∞ or 0.

∴ large damping for both lo + hi freq.

$$\rho = -\frac{1}{2}\cdot\frac{1}{C'L\omega^2}\left[(1 - L'C'\omega^2)(1 - LC\omega^2)\right]$$

If $L'C' > LC$, one factor is (−) while other is (+);
Hence ρ is positive.



Desirable for ρ not to be positive, so as to get

only one band; this can be done by making $L'C' = LC$.

then $\rho = -\frac{1}{C'L\omega^2}\left(1-\frac{\omega^2}{\omega_0^2}\right)^2$ is always negative.

then transmission band is in neighborhood of the resonant freq. $\omega_0^2 = \frac{1}{LC} = \frac{1}{L'C'}$;

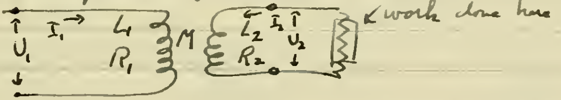
then $\sinh \gamma/2 = \frac{i}{\sqrt{2C'L\omega^2}}\left(1-\frac{\omega^2}{\omega_0^2}\right)$.

$$= \frac{i}{\sqrt{2\frac{L}{L'}}}\frac{\omega_0}{\omega}\left(1-\frac{\omega^2}{\omega_0^2}\right)$$

$$= i\sqrt{\frac{2L'}{L}}\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right).$$

For sharp resonance, i.e. narrow band ~~transmittance~~ transmittance, make $L' >> L$.

Back to transformer problem:



$$U_1 = I_1(R_1 + i\omega L_1) + i\omega M I_2$$

$$U_2 = I_2(R_2 + i\omega L_2) + i\omega M I_1$$

$$I_2 = \frac{U_1 - I_1(R_1 + i\omega L_1)}{i\omega M}.$$

$$U_2 = \frac{U_1 - I_1(R_1+i\omega L_1)}{i\omega M}(R_2 + i\omega L_2) + i\omega M I_1$$

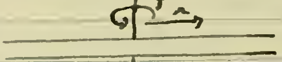
$$= U_1\frac{R_2 + i\omega L_2}{i\omega M} - \frac{(R_1+i\omega L_1)(R_2+i\omega L_2)+\omega^2 M^2}{i\omega M}I_1$$

~~look for sol'n : $= \lambda U_1$~~.

Suppose now R_1, R_2 are small, and coupling $k \cong 1$.
then second term of U_2 vanishes;
and
$$\frac{U_2}{U_1} = \frac{L_2}{M} = \sqrt{\frac{L_2}{L_1}} = \frac{n_2}{n_1}.$$
$$I_2 = \frac{U_1}{i\omega\sqrt{L_1L_2}} - I_1\sqrt{\frac{L_1}{L_2}}.$$
if $U_1 << I_1 L_1 \omega$,
$$-\frac{I_2}{I_1} = \sqrt{\frac{L_1}{L_2}} = \frac{n_1}{n_2}.$$
Justified if $n_2 >> n_1$.

oblem: / t other / dition

So far our circuits have been different from mechanical cases, in that we have always gotten just one (1 circuit) 2 (2 circuit) etc. characteristic frequencies; nothing like resonance to overtones or harmonics — this is because C, L, R are independent of ω. Actually both C & R depend considerably on ω; For R, known as "skin effect" - For C, potential is no longer constant over surface:



disc condenser.

Field intensity E_z - ordinarily the same throughout condenser. It has a time derivative:
$$\frac{dE_z}{dt} \rightarrow c \text{ curl } H \qquad \text{(Maxwell)}$$

H will be in φ direction H_φ, a function of r. For low ω: $H_\varphi \cdot 2\pi r = \int c\, \dot{D}_z$
$$= c\varepsilon i\omega E_z \pi r^2$$

$\therefore \quad H_\varphi = i\omega c \varepsilon \frac{1}{2} E_z$

But $\frac{\partial H_\varphi}{\partial t}$ will produce a field curl E, in φ direction; causes $\frac{\partial E_z}{\partial r} \sim \omega^2 E_z$.

Hence r-dependence of E_z will increase with ω^2. For low frequency, can neglect r-dependence; quasi-stationary state; no longer true for rapidly oscillating field.

$\therefore$ For large ω: must have $E_z(r)$ $H_\varphi(r)$.

Simplest case of resonating cavity; Useful for generation of ultra-high freq. waves.

Oct. 17, '41 [Digression on 1st problem: In calculating L, theoretical formula is $L = \frac{4\pi^2 a^2 n^2}{b}$ a = radius, b = length, n = no. of turns. Derived by line integral of H; assume no field outside — poor if of finite length. Correction $K(\frac{b}{2a})$ can be found tabulated — (is a complete elliptic integral). Formula without correction in our problem gives 24.7 μh. K = .90, whence L = 22.3 μH. Further corrections due to finite thickness; if c is depth of windings, subtract $4\pi \frac{n^2 a c}{b}(\log 2 + B(\frac{b}{c}))$ this gives 3.9 μH; $\therefore L = 18.4$ μh.

Ham handbook gives: $L = \frac{.2 A^2 n^2}{3A + 9B + 10C}$ A = diam., inches; B, length, inches; C, thickness, inches.

$= 18.7$ μh in this case.]

Problem 4: Want to transmit 1,000 kc: in fact want to transmit a band 1,000 ± 10 kc: 1) Find filter constants. [Smallest C: 100 cm.] 2) What is resistance of coils; — (again 1 mm Cu wire — 1 cm mean diameter of coil (Simple formula for L no corrections)] 3) Calculate performance at resonance. 4) 950 kc is to be damped out to 10^{-5} of original intensity; $\therefore$ how many filters;

At low ω, same E throughout condenser at a given time;
But for high ω, E takes some time to be propagated from center where it is first formed, to rim.

Assume circular condenser; then $E_z = E_z(r, \varphi)$.
[Here E_z will not depend on φ]
Will use wave equation [2nd order eq'n from Maxwell's eq'ns]

$$\nabla \times \underline{H} = \frac{\varepsilon}{c}\dot{\underline{E}}$$
$$\nabla \times \underline{E} = -\frac{1}{c}\dot{\underline{B}} = -\frac{\mu}{c}\dot{\underline{H}}$$

ε = dielectric const.
μ = permeability.
$\mu \cong 1$.

Whence $\nabla^2 \underline{E} = \frac{\varepsilon}{c^2}\ddot{\underline{E}}$ (div $\underline{E} = 0$ — no space charge)

gives waves of velocity $\frac{c}{\sqrt{\varepsilon}}$
where $\sqrt{\varepsilon} = n$ (refraction index).

Here $E_z = E_0 e^{i\omega t}$
$\ddot{E}_z = -\omega^2 E_z$; in cylindrical coordinates:
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

whence $\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} = -\frac{\omega^2 \varepsilon}{c^2} E$.

Can assume immediately that
$E \sim e^{im\varphi}$

Whence $\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} - \frac{m^2}{r^2}E + k^2 E = 0$

$k = \frac{\omega}{v} = \frac{\omega}{c/\sqrt{\varepsilon}} = \sqrt{\varepsilon}\,\frac{\omega}{c}$.

(k = no. of waves in 2π cms.)

"This is equation of Bessel: order m, argument kr."
Hence $E \sim J_m(kr)$. (other Bessel

f'ns blow up at origin, hence take only J)

Whence $E_z = a_m J_m(kr)\, e^{im\varphi} e^{i\omega t}$.
(In general a complex).

Now $J_m(kr) \sim (kr)^m$ for small r. ; Hence for m = 0, E = 0 where lead comes in! ; also, φ dependence requires E = 0 at center.

For Lead at center, m = 0,

and $E_z = a_0 J_0(kr)\, e^{i\omega t}$.

Oct. 20

Bessel F'n reference: Stratton, E.M. Theory

Want to find charge on condenser.
Charge density at surface = $\frac{\varepsilon E}{4\pi}$;

$$\text{Total charge} = \int_0^R \frac{\varepsilon E}{4\pi} \cdot 2\pi r\, dr$$

$$= \frac{\varepsilon}{2} \int_0^R E\, r\, dr$$

$$= \frac{\varepsilon a}{2} \int_0^R J_0(kr)\, r\, dr$$

For k very small, get static case;

$$J_0(kr) = 1$$

$$Q = \text{Total charge} = \frac{\varepsilon a}{2} \cdot \frac{R^2}{2}$$

For k not very small :

$$Q = \frac{\varepsilon a}{2} \frac{1}{k^2} \int_0^{kR} J_0(x)\, x\, dx$$

$$= \frac{\varepsilon a}{2k^2} \cdot kR\, J_1(kR)$$

$$= \frac{\varepsilon a}{2k} R\, J_1(kR).$$

∴ Ratio of capacity $k = k$ to cap. $k = 0$,

is : $\frac{C_k}{C_0} = \frac{2}{kR} J_1(kR)$.

(small k, $J_1(kR) \to \frac{kR}{2}$, and $\frac{C_k}{C_0} \to 1$).

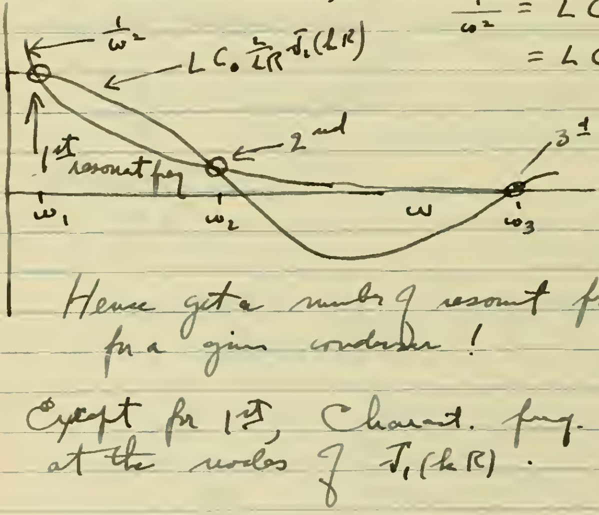
In general, $\frac{C_k}{C_0} < 1$.

For nodes of $J_1(kr)$, C_k vanishes.

Consider circuit containing this capacity & an inductance : L, C.

$\frac{1}{\omega^2} = LC$

$= LC_0 \frac{2}{kR} J_1(kR)$.



etc.

(assuming L indep. of ω)

Hence get a number of resonant freq. for a given condenser!

Except for 1st, Charact. freq. are very nearly at the nodes of $J_1(kR)$.

Zeros of J_1 : First zero ~ 4 Later on, near $\pi(n + 1/4)$

$kR \approx \pi(n + 1/4)$

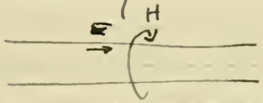
$\lambda = \frac{2\pi}{k} \approx \frac{2\pi R}{\pi(n+1/4)} = \frac{2R}{n+1/4}$. approx. n wave lengths to diameter,

etc.

So we get overtones. — but no simple connection between 1st & others — 1st one given by static formula $\frac{1}{2c}$, others depend on dimensions of conductor —

Skin effect:

frequency dependence of Resistance.

Conductor: 

$$\nabla \times \underline{H} = \frac{4\pi \dot{\underline{j}}}{c} + \frac{\epsilon}{c}\dot{\underline{E}}$$

$$\nabla \times \underline{E} = -\frac{\mu}{c}\dot{H}$$

Neglect $\dot{\underline{E}}$ in comp. with $\underline{j}$; $\quad \underline{j} = \sigma \underline{E}$

Then $\nabla \times \nabla \times E = -\frac{4\pi\mu}{c^2}\sigma \dot{E}$

if again $\nabla \cdot E = 0$: $\quad \nabla^2 E = \frac{4\pi\mu\sigma}{c^2}\dot{E}$

if $E \sim E e^{i\omega t}$

then $\nabla^2 E = \frac{4\pi\mu\sigma i\omega}{c^2} E$.

or $\nabla^2 E = -k^2 E$

$\sigma_{e.s.u.} : \left[\frac{j}{E}\right]$

$= \sigma_{e.m.u.} \cdot c^2$

$= \frac{c^2}{10^9}\sigma_{mho/cm}$

k^2 is pure imaginary:

$$k^2 = -\frac{2\pi\mu\sigma_{mho}}{10^9} \cdot \omega \cdot 2i$$

$$k = \sqrt{\frac{2\pi\mu\sigma}{10^9}\omega}\,(1 \mp i)$$

Symmetrical solution is again:
(solid wire) $\quad E = a\, J_0(kr)$.

For $kR \ll 1$, $J_0 = 1$, & E is constant throughout the wire; hence j constant throughout wire.

Take conjugate complex all along here
mistake in sign —

Otherwise J_0 increases with increasing absolute value of complex argument. $\underset{x>>1}{\rightarrow}$ $J_0(x) = \frac{1}{2}\sqrt{\frac{2}{\pi x}}\left[e^{i(x-\vartheta_1)} + e^{-i(x-\vartheta_1)}\right]$.

$k = k_0(1+i)$

$x = k_0 R(1+i)$

$e^{-i(x-\vartheta_1)} \rightarrow e^{-i(k_0 R(1+i)-\vartheta_1)}$

$= e^{k_0 R}$ ———

hence increases.

$\therefore$ j also is larger near surface — in fact practically all current at surface — so called skin effect.

Oct 22, '41 For Cu: $\sigma \cong 6 \cdot 10^5$ mho/cm. $\mu = 1$ $\omega = 2\pi f$

$$k_0 = \sqrt{\frac{2\pi\sigma\mu\omega}{10^9}}$$

$$= 2\pi\sqrt{\frac{6\cdot 10^5 \cdot f}{10^9}}$$

$$\cong 2\pi \cdot 10^{-2}\sqrt{6f}$$

$$\cong .15\sqrt{f} \text{ cm}^{-1}.$$

For A.C., $f = 60$, $k_0 \simeq 1$.

$\therefore k_0 r \sim 1$ for $r = 1$ cm.

i.e. ordinary A.C. will start to show skin effect for Cu conductors ~ 1 cm in radius.

For $f \sim 10^6$ $k_0 \sim 150$ cm^{-1}

Now even wire of .1 diameter ($r = .05$ cm)

$k_0 r = 7.5$ – large skin effect.

What to do about it? Can use a hollow conductor. But only for large diameters —

Or take large number of small wires But this isn't much improvement — especially if many, closely packed — Might arrange small wires in hollow ring thus: !

Total current: $I = \int_0^{r_0} j \; 2\pi r \, dr$

Compare with $I_1 = j(r_0) \cdot \pi r_0^2$ — current at surface of wire.

$$\frac{I}{I_1} = \require{cancel}\cancel{\frac{R_\omega}{R_0}} \; \frac{R_0}{R_\omega}$$

$j = J_0(hr)$ $\qquad \therefore \; I = 2\pi \int_0^{r_0} J_0(hr)\, r\, dr$

$$= \frac{2\pi}{h} r_0 J_1(hr_0)$$

$$\therefore \; \frac{I}{I_1} = \frac{\frac{2\pi}{h} r_0 J_1(hr_0)}{\pi r_0^2 J_0(hr_0)}$$

$$= \frac{2}{hr_0} \frac{J_1(hr_0)}{J_0(hr_0)} = \frac{R_0}{R_\omega}.$$

a) if $|hr_0| << 1$: $J_1 = \frac{1}{2} h r_0$

$J_0 = 1$

$\therefore R_\omega \to R_0$.

Or expand further & get idea when effect starts.

b) $|hr_0| >> 1$:

$$J_1 = \frac{1}{2}\sqrt{\frac{2}{\pi x}} \; e^{i^{3/2} x + i \frac{3}{4}\pi} \ldots$$

$$J_0 = \frac{1}{2}\sqrt{\frac{2}{\pi x}} \; e^{i^{3/2} x - i \pi/4} \ldots$$

$$\frac{2 J_1}{J_0} = \frac{2}{x} i$$

$$x = h_0 (1 \mp i) r_0$$

$$\frac{R_\omega}{R_0} \to \frac{h_0 r_0}{1 \mp i} = \frac{1}{2} h_0 r_0 (1+i)$$

$$\left|\frac{R_\omega}{R_0}\right| >> 1.$$

Also R_ω is now complex : effective inductance equal in magnitude to resistance -
[using previous example, $\left(\frac{R_\omega}{R_0}\right)_{real} = \frac{1}{2} k r_0 \sim 4$ -

$$R_\omega = \tfrac{1}{2} R_0 \, k_0 r_0$$

$$R_0 \sim \frac{1}{\sigma r_0^2} \quad , \quad k_0 \sim \sqrt{\sigma}$$

$$\therefore R_\omega \sim \frac{\sqrt{\sigma} r_0}{\sigma r_0^2} \sim \frac{1}{\sqrt{\sigma} r_0^2} .$$

$\therefore$ R_ω still depends on σ & r_0 in same way as R_0, but only as square root —

All this neglected term $-\frac{\mu \varepsilon \omega^2}{c^2} E$ in wave eq'n
this will start to come in at $\omega \gtrsim 10^{19}$ - because $\varepsilon \sim -10^5$ in metals on account of free electrons.

Limiting skin effect case (thick conductor or high freq) can be treated simply, without Bessel f'n's.

Oct. 24 Cavity of cylindrical shape:
problem. Excite at distance r_1 from center:

Non symmetrical oscillation
1) What is frequency?
2) If not closed, what is effective capacity?

$k \sim \sqrt{\sigma \mu \omega}$ For ferromagnetics, thinner skin.
$\underline{B} = \mu \underline{H}$ usually holds for $B < 10^4$ gauss.

For long straight wire, $H = \frac{2 I_{amp}}{r}$ if $r = r_{max}$,

$H = 2\, I_{amp.}$ ∴ $B = 2\mu$ for $I = 1$ amp.
So for most materials formula is valid; not for very high permeabilities, e.g.: permalloy.

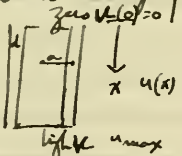
Coil at high freq.

Main disturbing effect is distributed capacity.
Hard to calculate; not too hard to estimate.

Consider coil of two layers; (in & out at same end)

Greatest pot. diff at end;

We have a sort of cylindrical condenser, with varying pot. diff.



(radius >> separation)

Cap./unit length: $C = \frac{2\pi a}{4\pi d} = \frac{a}{2d}$.

Q: charge/cm $= C\, u(x)$

~~$i = \frac{\partial Q}{\partial t}$~~

$$\frac{\partial Q}{\partial t} = -\operatorname{div} I = -\frac{\partial I}{\partial x} = C\frac{\partial u}{\partial t}.$$

$$\text{Field} = -\frac{\partial u}{\partial x} = R\, I + L\frac{dI}{dt}.$$ R = res./cm. L = ind./cm.

Eliminate & get:

$$\frac{\partial^2 u}{\partial x^2} = -R\frac{\partial I}{\partial x} + LC\frac{\partial^2 u}{\partial t^2}$$
$$= RC\frac{\partial u}{\partial t} + LC\frac{\partial^2 u}{\partial t^2}.$$

if R is small; then

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2}$$

$$u \sim e^{i\omega t}\sin(\sqrt{LC}\,\omega x)$$

$u(0) = 0$ $\qquad I \sim \cos(\sqrt{LC}\,\omega x)$

∴ Current in coil is smaller (for short coil) at leads than at middle & top! (made up in charging condenser) If length of coil is b, then for $\sqrt{LC}\,\omega b = \pi/2$, no current at all!

critical frequency: $\omega = \frac{\pi}{2b\sqrt{LC}}$.

L, C are cap & ind./unit length;

∴ $\omega = \frac{\pi}{2\sqrt{L'C'}}$ L', C', total ind., cap.

Estimate: $L' = \frac{4\pi^2 a^2 n^2}{b}$ $C' = \frac{ab}{2d\,c^2}$.

$$\sqrt{L'C'} = \sqrt{\frac{4\pi^2 a^2 n^2 ab}{2b\,d\,c^2}}$$

$$= \frac{2\pi a n}{c}\sqrt{\frac{a}{2d}}.$$

$$\lambda_{crit.} = \frac{2\pi c}{\omega} = \frac{2\pi c\cdot 2\sqrt{L'C'}}{\pi}$$

$$= 4c\cdot\frac{2\pi a n}{c}\sqrt{\frac{a}{2d}}$$

$$= 4\sqrt{2}\,\pi a n\sqrt{\frac{a}{d}}.$$

$$= 2\pi a n\sqrt{\frac{8a}{d}}.$$

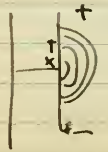
$2\pi a n \approx$ length of wire in coil.

∴ make length of wire small comp. with desired wave length.

Single ~~[illegible]~~ coil: $q = 2\pi a \cdot$ charge/unit area (charge/cm.)

$= 2\pi a\cdot\frac{1}{4\pi}\cdot E$ $E = \frac{2u}{\pi x}$!

$= \frac{a}{2}\cdot\frac{2u}{\pi x}$ pot. diff divided by length of field lines!



lines of force.

again $-\frac{\partial I}{\partial x} = \frac{a}{\pi x}\frac{\partial u}{\partial t}$

$-\frac{\partial u}{\partial x} = L\frac{\partial I}{\partial t}$. if only inductance

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{aL}{\pi x} \frac{\partial^2 u}{\partial t^2}$$

or, in conventional units: $\frac{\partial^2 u}{\partial x^2} = \frac{aL}{\pi x c^2} \frac{\partial^2 u}{\partial t^2}$.

$$u \sim \sqrt{x}\, J_1\left(\sqrt{\tfrac{4}{\pi} L a x}\ \tfrac{\omega}{c}\right)$$

$$\lambda_a \sim \sqrt{L' a} \qquad L' = Lb.$$

Oct. 27 $J_1(y) = \frac{1}{2} y$ for small y.

$\therefore$ for small x, $u \sim \sqrt{x} \cdot \frac{1}{2}\sqrt{\frac{4}{\pi} L a x \frac{\omega^2}{c^2}}$

$\sim x \cdot \sqrt{K}$.

Let $K = \frac{L a \omega^2}{\pi c^2}$

For large ω $\frac{J_1}{\frac{1}{2}y} < 1$

Get I from second eq'n:

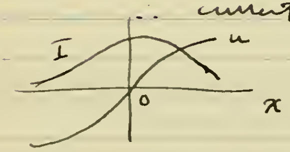
$$\frac{\partial u}{\partial x} = -iL\omega I = \frac{1}{\sqrt{2K}} \frac{d}{dx}\left(y J_1(y)\right)$$

$$y = \sqrt{2Kx}$$

$$\frac{1}{2K}\frac{d}{dx} = \frac{1}{2y}\frac{d}{dy}$$

$$\therefore -Li\omega I = \sqrt{\tfrac{K}{2}}\, J_0(y).$$

$\therefore$ current is again distributed like J_0



[in double layer case, we found $u \sim \sin x$, $I \sim \cos x$, same general appearance].

At first node of J_0: $y = 2.4 = 2\sqrt{Kx}$

if length of coil is b, then u vanishes for

$$K b/2 = \frac{(2.4)^2}{4} = 1.4$$

$$K = \frac{1.4}{b/2} = \frac{2.8}{b}.$$

Whence $\frac{\omega_{crit}^2}{c^2} = \frac{2\cdot 8\cdot \pi}{Lab}$

$$\therefore \frac{\omega_{crit}}{c} = \sqrt{\frac{2\,8\pi}{Lab}}$$

and $\lambda_{crit} = \frac{2\pi c}{\omega_{crit}} \simeq 2\sqrt{Lab}$.

$= 2\sqrt{L'\,a}$ L' = total inductance

$= 2\sqrt{\frac{4\pi^2 n^2 a^2}{b}\cdot a}$

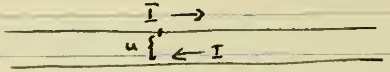
$= 2\pi n a \cdot \sqrt{4\frac{a}{b}}$. $b \gg a$ probably?

$2\pi n a$ = length of wire;

For double layer, have $\lambda_{crit} = 2\pi n a \sqrt{8\frac{a}{d}}$ where $d \ll a$.

Hence λ_{crit} for double layer much greater than for single layer.
$\therefore$ single layer coil better at high frequencies.

PROPAGATION OF ELECTRIC WAVES ALONG WIRES.



Two wires; at each pt. a diff. of pot. u between wires, current I in wires.

Even at low freq., current not the same everywhere.
Both u & I are functions of both x & t.

Consider a cm. of length; u changes by $\frac{du}{dx}$; this is accounted for by $RI + L\frac{dI}{dt}$: $-\frac{du}{dx} = RI + L\frac{\partial I}{\partial t}$ $R, L,$ per unit length

$-\frac{\partial I}{\partial x} = \frac{\partial q}{\partial t} = C\frac{\partial u}{\partial t} + Gu$ C "

G accounts for leakage, corona, etc. ← to make it symmetrical

Oct. 29, 1941 Equations of ~~transmission~~ line:

$$\begin{cases} -\frac{\partial I}{\partial x} = C\frac{\partial u}{\partial t} + G u \\ -\frac{\partial u}{\partial x} = L\frac{\partial I}{\partial t} + R I \end{cases}$$

all quantities per unit length; usually convenient to take a large unit, as e.g. km.

Parallel wires: $C = l_1 \frac{1}{4}\left(\ln\frac{a}{r}\right)^{-1}$ $\qquad$ l_1 = length of unit (e.g. 10^5)

$L = 4 l_1 \left(\ln\frac{a}{r}\right)$

r = radius of wire

a = distance between wires.

Concentric cable: $C = l_1\left(2\ln\frac{r_1}{r_2}\right)^{-1}$

$L = l_1 \cdot 2\ln\frac{r_1}{r_2}$

$C > l_1$, $L < l_1$ for concentric cable } roughly.
$C < l_1$, $L > l_1$ " parallel wires }

Hence per km.: $C \sim 10^5\,\mathrm{cm} = 0.1\,\mu F$

$L \sim 10^5\,\mathrm{cm} = 0.1\,\mathrm{mH}$.

for Cu: $R = 1.7\cdot 10^{-6}\cdot\frac{10^5}{.031} \sim 5$ ohm/km

if $d \sim .2$ cm. Area $= \pi \times .04$

$G \sim 10^{-6}$ mho/km (Vilbig)

[If G is to be this, then conductivity of insulating material should be $\leq 10^{-11}$ mho/cm].

Assume $u = u(x)e^{i\omega t}$

$I = I(x)\,e^{i\omega t}$

then $-\frac{\partial I}{\partial x} = u(G + iC\omega)$

$-\frac{\partial u}{\partial x} = I(R + iL\omega)$

Combining : $\frac{\partial^2 u}{\partial x^2} = u(R+iL\omega)(G+iC\omega)$

$\therefore u \sim e^{kx}$

where $k^2 = (R+iL\omega)(G+iC\omega)$

k is wave no. in km^{-1} ; x measured in km.

$k^2 = (GR - CL\omega^2) + i\omega(RC + GL)$.

$k = i\alpha + \beta$

$-\alpha^2 + \beta^2 = -CL\omega^2 + GR$

$2\alpha\beta = \omega(RC + GL)$

Want a line with $GR << CL\omega^2$;

hence $\beta << \alpha$

$\alpha \simeq \omega\sqrt{CL}$

$\beta \simeq \frac{\omega(RC+GL)}{2\omega\sqrt{CL}} = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$

$\therefore u_1 \sim e^{i\alpha x + \beta x}$ or $e^{-i\alpha x - \beta x} = u_2$.

In general, $u = a_1 u_1 + a_2 u_2$.

In long line, only exp. decreasing sol'n will be important — (except at end). Hence

$u = e^{-i\alpha x - \beta x} \cdot e^{i\omega t}$.

$= e^{i(\omega t - \alpha x)} \cdot e^{-\beta x}$

damped travelling wave ; Wave velocity $= \frac{\omega}{\alpha} \simeq \frac{1}{\sqrt{CL}}$.

$CL \simeq \frac{\ell_1^2}{c^2}$;

$\therefore$ Wave velocity $\simeq \left(\frac{c}{\ell_1}\right)$ km/sec.

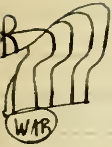
In cable with dielectric : $v \sim \frac{c}{\sqrt{\varepsilon}}$.

$\therefore$ Get running waves travelling with almost the velocity of light ; damped by factor $e^{-\beta x}$.

Damping per km : $\beta = \frac{1}{2}\left(\frac{R}{\sqrt{L/C}} + G\sqrt{L/C}\right)$ $\qquad R \sim 5$, $\sqrt{L/C} \sim 30$, $G \sim 10^{-6}$

$\sim \frac{1}{2}\left(\frac{5}{30} + 30 \cdot 10^{-6}\right) \sim \frac{1}{12}$.

Damping almost entirely due to ohmic resistance.



CDEFGHIJKLMNO

QRSTUVWXYZ

Wire:	Diam. wire:	G	L (mh/km)	C (μF/km)	R
Cu.	1.5 mm	10^{-6}	2.3	.0051	3.1
	5 mm.	10^{-6}	1.8	.0068	1.9
Cable	.6 mm	10^{-2}	.7	.031	130
	2.0 mm	10^{-6}	.6	.041	12

More damping in cable — $\frac{R}{\sqrt{L/C}} \sim 1/3$

For most favorable case, can get $R/\sqrt{L/C} \sim \frac{1}{250}$

i.e. 500 km to drop to 1/e of value.

Oct 31.

	Diam.	[illegible]	$\frac{1}{2} R\sqrt{\frac{C}{L}} \leftarrow \beta \rightarrow \frac{1}{2} G\sqrt{\frac{L}{C}}$		Ratio	$1/\beta$ km
Free air line	1.5 mm	2.93	.0157	.00033	50	62
	5 mm.	2.87	.00186	.00025	7.5	475
Cable	.6 ..	2.15	.43	.00007	6,000	2.3
	2.0 -	2.02	.044	.00006	800	20

We have assumed above: $L\omega \gg R$ i.e. $\omega \gg \omega_1 = R/L$.

$C\omega \gg G$ .. $\omega \gg \omega_2 = G/C$.

For above cases:

	$\frac{\omega_1}{2\pi}$ (kc)	$\frac{\omega_2}{2\pi}$ (kc)
	2.2	.03
	.17	.025
	30	.005
	3.1	.004

∴ $f_1 = \frac{\omega_1}{2\pi}$ is about in region of telephone transmission.

Damping needs to be reduced — esp. for cables —

Damping is given by $R\sqrt{\frac{C}{L}}$; R can't be reduced; C can't be reduced – esp. if a cable – which is desirable – is used. Hence hope to increase L.
Pupin – put in a coil from time to time to do this.

If we have Pupin coils of L_p and R_p placed distance s apart; Additional L is L_p/s, R_p/s

so that $\beta = \frac{1}{2}\left(R + \frac{R_p}{s}\right)\sqrt{\frac{C}{L+L_p/s}} + G\sqrt{\frac{L+L_p/s}{C}}$.

For low freq. Neglecting R_p/s, Minimum damping would be for two terms equal i.e. $R\sqrt{\frac{C}{L'}} = G\sqrt{\frac{L'}{C}}$

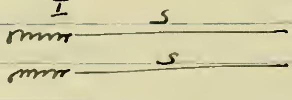
this gives $L' = \frac{RC}{G} = 4$ henry/km too much!

This will also behave like a filter – critical frequency

$\omega_0 \sim \frac{1}{\sqrt{(Ls+L_p)Cs}}$ so that increasing s will lower ω_0
also decreasing L_p " " "

So there is another limitation on pupin coils –

theory:



P.d. decrease $\Delta u = -I\mathfrak{R}_p$ in coil

In line $-\frac{\partial u}{\partial x} = (R + iL\omega)I$ $\mathfrak{R}_p = R_p + i\omega L_p$.

$-\frac{\partial I}{\partial x} = (G + iC\omega)u$

then $I = a e^{hx/s} + b e^{-hx/s}$

$h = s\sqrt{(R+iL\omega)(G+iC\omega)}$

If at $x=0$ $I=1$, $a+b=1$.

$$u = -Zae^{hx/s} + Zbe^{-hx/s} \qquad Z = \text{wave resistance}$$

for $u \sim e^{hx/s}$, $-\frac{\partial u}{\partial x} = -\frac{h}{s}e^{\frac{h}{s}x} = -\sqrt{}\; u = (R+iL\omega)I$

$$\frac{u}{I} = -\sqrt{\frac{R+iL\omega}{G+iC\omega}} = -Z.$$

$\frac{u}{I}$ at left $= V$

at right of coil $= V - R_p$

$\therefore$ if $I=1$, get a condition:

$$Z(b-a) = V - R_p.$$

$$\therefore a = \frac{Z - V + R_p}{2Z}$$

$$b = \frac{Z+V-R_p}{2Z}.$$

at left end of next coil, again $\frac{u}{I}(s) = V$

& so on $\therefore$ for $x=s$

$$\frac{u}{I} = a: -Zae^{h} + Zbe^{-h}$$

$$V(ae^{h}+be^{-h}) = Z(-ae^{h}+be^{-h})$$

Nov. 3 $\quad (Z-V)(Z+V-R_p)e^{-h} = (Z+V)(Z-V+R_p)e^{h}$

$$(Z^2-V^2)(e^{-h}-e^{h}) = R_p[Z(e^{h}+e^{-h}) + V(e^{h}-e^{-h})]$$

$$\therefore -Z^2+V^2-VR_p - ZR_p\coth h = 0$$

$$\therefore V = \frac{R_p}{2} \pm \sqrt{\tfrac{1}{4}R_p^2 + Z^2 + ZR_p\coth h}$$

if define wave resistance : $\frac{u\,(\text{center of coil})}{I\,(\quad''\quad)} = V - \frac{1}{2}R_p$

$$= \pm\sqrt{\tfrac{1}{4}R_p^2 + Z^2 + ZR_p\coth h}$$

Want to know $\lambda = \frac{I(s)}{I(0)}$.

$$= ae^{h} + be^{-h}$$

$$(Z-V)be^{-h} = (Z+V)ae^{h}$$

$$\therefore \lambda = ae^{h}\frac{2Z}{Z-V} = e^{h}\frac{Z-V+R_p}{Z-V}.$$

Expect $\lambda_1 \lambda_2 = 1$ (i.e. $\lambda_1 = e^{g}$ $\lambda_2 = e^{-g}$).

$$\lambda_1 \lambda_2 = e^{2h}\left(1 + \frac{R_p}{z - v_1}\right)\left(1 + \frac{R_p}{z - v_2}\right)$$

$$= e^{2h}\left[\frac{(z - v_1 + R_p)(z - v_2 + R_p)}{(z - v_1)(z - v_2)}\right]$$

But $v_1 + v_2 = R_p$

$v_1 v_2 = -z^2 - zR_p \coth h$ from quadratic eqn

$$\therefore (z - v_1)(z - v_2) = z^2 - zR_p - z^2 - zR_p \coth h$$

$$= -zR_p\left[1 + \coth h\right].$$

and $(z + R_p - v_1)(z + R_p - v_2)$

$$= (z + R_p)^2 - (z + R_p)R_p - z^2 - zR_p \coth h$$

$$= zR_p(1 - \coth h).$$

$$\therefore \lambda_1 \lambda_2 = e^{2h}\frac{zR_p(1 - \coth h)}{-zR_p(1 + \coth h)}$$

$$= -e^{2h}\left[\frac{e^{h} - e^{-h} - e^{h} - e^{-h}}{e^{h} - e^{-h} + e^{h} + e^{-h}}\right]$$

$$= -e^{2h} \cdot \frac{-2e^{-h}}{2e^{h}} = 1$$

$$\tfrac{1}{2}(\lambda_1 + \lambda_2) = \cosh g = \tfrac{1}{2}e^{h}\left[1 + \frac{R_p}{z - v_1} + 1 + \frac{R_p}{z - v_2}\right]$$

$$= e^{h}\left[1 + \tfrac{1}{2}R_p\frac{2z - v_1 - v_2}{(z - v_1)(z - v_2)}\right]$$

$$= e^{h}\left[1 + \tfrac{1}{2}R_p\frac{2z - R_p}{-zR_p(1 + \coth h)}\right]$$

$$= e^{h}\left[1 - \frac{R_p}{2}\cdot\frac{2Z - R_p \sinh h}{2R_p(\cosh h + \sinh h)}\right]$$

$$= e^{h}\left[1 - \frac{1}{2}\,\frac{2Z - R_p \sinh h}{Z\,e^{h}}\right]$$

$$= \left[e^{h} - \tfrac{1}{2}\left(2 - \frac{R_p}{Z}\right)\sinh h\right]$$

$$= \left[e^{h} - \sinh h + \frac{R_p}{2Z}\sinh h\right]$$

$$= \left[\cosh h + \frac{R_p}{2Z}\sinh h\right]$$

$$\therefore \cosh g = \cosh h + \frac{R_p}{2Z}\sinh h .$$

Z is almost real;

$$Z = \sqrt{\frac{L}{C}}\left(1 + \frac{R}{2i\omega L} - \frac{G}{2i\omega C}\right) \quad \text{for high } \omega.$$

h almost imaginary:

$$h = i\omega\sqrt{LC}\; s\left(1 + \frac{R}{2i\omega L} + \frac{G}{2i\omega C}\right).$$

$R_p = R_p + i\omega L_p \qquad L \gg R \qquad \therefore R_p$ almost imag.

$\therefore$ if $g = ia + b$

$\cosh g = \cos a \cosh b + i \sin a \sinh b$

expect b small : $\cosh b \sim 1$, $\sinh b \sim b$

if $h = i\alpha + \beta$ $\qquad \dfrac{R_p}{2Z} = X + iY$

$\cosh h = \cos\alpha\cosh\beta + i\sin\alpha\sinh\beta$

$\sinh h = i\sin\alpha\cosh\beta + \cos\alpha\sinh\beta$

$\therefore \cos a + i\,b\sin a = (i\sin\alpha\cosh\beta + \cos\alpha\sinh\beta)(X + iY)$

$\therefore \cos a = \cos\alpha - \Upsilon \sin\alpha$ neglect $\bar{X}\beta$
& $b \sin a = \beta \sin\alpha + \beta\Upsilon\cos\alpha + \bar{X}\sin\alpha$

$$\therefore \cos a = \cos(\omega\sqrt{LC}\, s) - \frac{\omega L_p}{2\sqrt{L/C}} \sin(\omega\sqrt{LC}\, s) .$$

$a > \alpha$, since $\cos a < \cos\alpha$. i.e. phase shift

$$2\sin^2 a/2 = 1 - \cos a = 2\sin^2 \alpha/2 + \frac{\omega L_p}{\sqrt{L/C}} \sin\alpha/2 \cos\alpha/2$$

$$= \sin\alpha/2 \left[2\sin\alpha/2 + \frac{\omega L_p}{\sqrt{L/C}} \cos\alpha/2 \right]$$

Now $L_p >> L \cdot s$

$$\alpha = s \cdot \omega\sqrt{LC} \quad \text{and} \quad \frac{\omega L_p}{\sqrt{L/C}} = \frac{\omega L_p}{L_s}\sqrt{LC}\cdot s = \alpha\frac{L_p}{L_s}$$

$$\therefore \frac{\omega L_p}{\sqrt{L/C}} >> \alpha .$$

Hence if a is to be $< \pi$, α must be very small (cmp. w.r.

Hence can expand in α:

$$\therefore 2\sin^2 a/2 = \alpha/2\left[\alpha + \frac{\omega L_p}{\sqrt{L/C}}\right] = \frac{\alpha^2}{2}\left[1 + \frac{L_p}{L_s}\right]$$

$$\therefore \sin^2 a/2 = \frac{\alpha^2}{4}\left[\frac{L_s + L_p}{L_s}\right]$$

$$= \frac{1}{4}\omega^2 s^2 LC\left[\frac{L_s + L_p}{L_s}\right]$$

$$\sin a/2 = \frac{\omega}{2}\sqrt{sC(L_s + L_p)} = \omega/\omega_0$$

$$\omega_0 = \frac{2}{\sqrt{sC(sL + L_p)}}$$

Nov 5, 1941 If higher powers of α are taken into account:

$$2\sin^2 a/2 = \frac{\alpha}{2}\left(1-\frac{1}{6}\frac{\alpha^2}{4}\right)\left[\alpha + \frac{\omega L_P}{\sqrt{L/C}}\left(1-\frac{\alpha^2}{8}\right)\right]$$

$\therefore$ multiplied by factor $\left(1-\frac{1}{6}\alpha^2/4\right)\left(1-\alpha^2/8\right) = \left(1-1/6\,\alpha^2\right)$

$\therefore$ 2nd approx: $\sin a/2 = \frac{\omega}{\omega_0}\left(1-\frac{1}{12}\omega^2 LCs\right)$

$LCs^2 \approx \frac{4}{\omega_0^2}\frac{Ls}{L_P}$ $\therefore \sin a/2 = \frac{\omega}{\omega_0}\left(1-\frac{1}{3}\frac{Ls}{L_P}\cdot\frac{\omega^2}{\omega_0^2}\right)$

(again if $L_P \gg Ls$, correction negligible).

Actually $\frac{L_P}{Ls} \approx 30$. ; $\therefore$ correction ~ .01 for $\omega \approx \omega$

$a \gg \alpha$; for $\omega \ll \omega_0$, ~~a~~ $\sin a/2 \sim a/2 = \omega/\omega_0$

$$a = \frac{2\omega}{\omega_0} = \sqrt{Cs(Ls+L_P)}$$

whereas $\alpha = \sqrt{CsLs}\,\omega$

$$\therefore a/\alpha = \sqrt{\frac{Cs(Ls+L_P)}{CsLs}} = \frac{2}{\omega_0}\cdot\frac{\omega_0}{2}\sqrt{\frac{L_P}{Ls}}$$

$$= \sqrt{\frac{Ls+L_P}{Ls}} \approx 5 \text{ or } 6.$$

Velocity $c' = \frac{\omega s}{a}$

Flat line: $c' = \frac{1}{\sqrt{CL}}$ ~ 200,000 km/sec

Pupin line: $c' = \frac{1}{\sqrt{C(L+\frac{L_P}{s})}}$ ~ 25,000 km/sec

much reduced.

Larger ω $a > \frac{2\omega}{\omega_0}$, $c'_{\text{hi freq.}} < c'_{\text{low freq.}}$

i.e. dispersion.

Bad: hi freq. arrive later than low freq! So insert a filter-like element which compensates - i.e. delays low freq.

$b \ll \beta$, where: $b\sin a = \beta\sin\alpha + \beta\,Y\cos\alpha + \bar{X}\sin\alpha$

$$\beta = \tfrac{1}{2}\left(R\sqrt{\tfrac{C}{L}} + G\sqrt{\tfrac{L}{C}}\right) \quad Y = \tfrac{1}{2}\omega L_P\sqrt{\tfrac{C}{L}}$$

$$\bar{X} = \tfrac{1}{2}L_P\sqrt{\tfrac{C}{L}}\left(\tfrac{R_P}{L_P} - \tfrac{1}{2}\tfrac{R}{L} + \tfrac{1}{2}\tfrac{G}{C}\right)$$

$$b\sin a = \tfrac{1}{2}\left(R\sqrt{\tfrac{C}{L}} + G\sqrt{\tfrac{L}{C}}\right)\left[s\omega\sqrt{LC} + \tfrac{s}{2}\omega L_P\sqrt{\tfrac{C}{L}}\right] + \tfrac{1}{2}L_P\sqrt{\tfrac{C}{L}}\left(\tfrac{R_P}{L_P} - \tfrac{1}{2}\tfrac{R}{L} + \tfrac{1}{2}\tfrac{G}{C}\right)\cdot s\omega\sqrt{LC}$$

AB. $= \frac{1}{2}(RC + GL)s^2\omega + \frac{1}{2}(R_pC + GL_p)s\omega$

$= \frac{1}{2}s\omega\left[C(Rs+R_p) + G(Ls+L_p)\right]$.

$\therefore b = \frac{C}{\sin\alpha} = \frac{C}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\frac{1}{2}s\omega\left[CR + GL\right]}{2\cdot\frac{\omega}{\omega_0}\cdot\sqrt{1-\frac{\omega^2}{\omega_0^2}}}$

$= \frac{1}{4}\frac{s\omega_0}{\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\quad\right]$

$\omega_0 = \frac{2}{\sqrt{Cs(Ls+L_p)}}$

~~$\therefore\ b/s = \frac{1}{2}\sqrt{\frac{1}{Cs(Ls+L_p)(1-\frac{\omega^2}{\omega_0^2})}}\left[\quad\right]$~~

~~$= \frac{1}{2}\frac{1}{\sqrt{Cs(Ls+L_p)(}}$~~

$b/s = \frac{1}{4}\sqrt{\frac{\omega_0^2}{1-\frac{\omega^2}{\omega_0^2}}}\left[\quad\right]$

$= \frac{1}{2\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\frac{C(R_s+R_p)}{\sqrt{Cs(Ls+L_p)}} + \frac{G(Ls+L_p)}{\sqrt{Cs(Ls+L_p)}}\right]$

$= \frac{1}{2\sqrt{1-\frac{\omega^2}{\omega_0^2}}}\left[\left(R+\frac{R_p}{s}\right)\sqrt{\frac{C}{L+L_p/s}} + G\sqrt{\frac{L+L_p/s}{C}}\right]$

Just as would be expected by simple theory —

Frequencies near $\omega = \omega_0$ damped considerably — up to $.8\omega_0$, OK.

Nov. 7, 1941

.A.

In order to transmit up to 7200 sec⁻¹:

If $C = 3.8\cdot10^{-8}$ F/km (1.4 mm diameter cable),

If $s = 1.7$ km, then $L_p = .02$ Henry

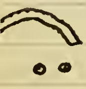
[$L\cdot s = .0012$ "].

this gives damping $b/s = .0186$ /km

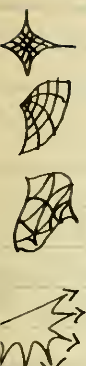
[flat line $\beta/s = .0460$ /km

$c' \sim$ 48,000 ~~20,000~~ km/sec

Wave resistance is increased.

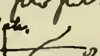


Can use a ferromagnetic substance; then inductance is increased by factor μ; damping reduced by factor $\sqrt{\mu}$. e.g. Can wind iron wire around cable. Must have high μ, but small hysteresis loss: Permalloy works well

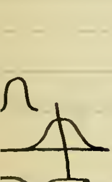


Principle of permalloy: It is known that hysteresis originates in the magnetic domains which are under strain, which gives preferred direction for magnetization; Hence to magnetize in a certain direction, must turn the domains, which requires much work; also when demagnetized, they don't return to original positions. So: small μ, large hysteresis. So to get good ferromagnetic, need one without strains; none such — so next best, one where strains don't influence magnetization.

Consider iron (Fe) lump ←▭→; if stretched in a certain direction, preferred magnetization direction is in direction of stretch; for Ni, preferred direction is opposite to direction of stretch [magnetostriction] [Converse also holds - i.e. magnetization in direction causes stretch ∥ or ⟂ to direction] Now make an alloy of Fe + Ni (which is possible in any proportion); Plot magnetostriction vs. composition;



it crosses axis once - at that place, no magnetostriction, high μ, + low hysteresis.



Using this stuff, can get good transmission up to high frequencies & reduced damping.

So can use carrier frequencies modulated by sound, instead of ~~transmitting~~ audio directly —

e.g. If frequencies up to 5000 are to be transmitted, can modulate a 50,000 sec^{-1}, say, wave by them; then have 45–55 kc. waves, found at end with a filter, or by beating with 50 kc oscillator; can do this also with carrier frequencies 40 & 60 kc; & so ~~transmit~~ many at once.

Low frequencies.

We had always before: $R \ll L\omega$; OK for up $> .5$ to 5 kc (flat [illegible]

$f > \sim 200$ sec^{-1} ([illegible]

But not justified for 60 cycle transmission; even if there were increased L, which there isn't —

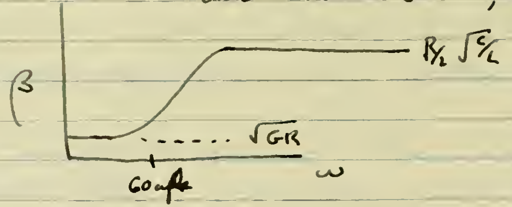
$$k = \sqrt{(R+iL\omega)(G+iC\omega)} \qquad G \ll C\omega \text{ even for 60 sec}^{-1}$$

if $L\omega \ll R$, $k \approx \sqrt{i}\sqrt{CR\omega} = \sqrt{\frac{CR\omega}{2}}(1+i)$

$k_{real} \cong k_{im.}$; $\therefore$ as much damping as phase shift & decrease with decreasing frequency.

If ω so small that $G \gg C\omega$,

then $k = \sqrt{GR}$, real.



Nov. 10 '41 In actual line, amplifiers are put in after a decrease of about e^{-3} in intensity of signal

$$c' = \frac{\omega}{\alpha} \; ; \quad \alpha = \text{I.P.}\sqrt{(R+iL\omega)(G+iC\omega)}$$

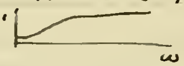
High freq: $\alpha = \omega\sqrt{LC}$, c' indep. of ω.

$$c' = \frac{1}{\sqrt{LC}}$$

Medium freq: $\alpha \sim \omega^{1/2}$ $\therefore c' \sim \omega^{1/2}$.

$\therefore c'$ decreases w. decreasing freq.

Low freq: $\omega < \theta/c$, $\alpha \sim \omega$ $\therefore c' =$ const. again.

$\therefore c'$ behaves about like β: 

Behavior of current & potential at end of line.

Put impedance R_E at end of line.

At end: $u = I R_E$ $(x = x_0)$.

Along line: $I = a e^{kx} + b e^{-kx}$ $k = \sqrt{(R+iL\omega)(G+iC\omega)}$

$u = -Z a e^{kx} + Z b e^{-kx}$ Z - wave resistance $= \sqrt{\dfrac{R+iL\omega}{G+iC\omega}}$.

$$\therefore -Z a e^{kx_0} + Z b e^{-kx_0} = R_E(a e^{kx_0} + b e^{-kx_0})$$

$$a e^{kx_0}[-Z - R_E] = b e^{-kx_0}[R_E - Z$$

$$\therefore \frac{a e^{kx_0}}{b e^{-kx_0}} = \frac{Z - R_E}{Z + R_E}.$$

1) Line with open end: $R_E = \infty$,

$$\therefore a e^{kx_0} = -b e^{-kx_0} = -A$$

$$\therefore I = A e^{k(x_0 - x)} - A e^{k(x - x_0)}$$

$$= 2A \sinh k(x_0 - x).$$

Node at $x = x_0$, as it should.

Standing waves, with amplitude decreasing from beginning of line to end. If $k = i\alpha + \beta$,

then $\sinh k(x_0 - x) = \sinh\beta(x_0 - x)\cos\alpha(x_0 - x)$
$+ i\cosh\beta(x_0 - x)\sin\alpha(x_0 - x)$

For high freq: $\alpha \gg \beta$; several periods (back from end of line) before $\sinh\beta$ is as big as cosh;

$$\therefore I = 2iA\sin\alpha(x_0 - x)$$

Going on toward beginning, amplitude increases almost exponentially; both terms equal

Maximum of $|I|$: $|I|^2 = \sinh^2\beta\cos^2\alpha + \cosh^2\beta\sin^2\alpha$

$= \sinh^2 - \sinh^2\sin^2 + \cosh^2\sin^2$

$= \sinh^2\beta + \sin^2\alpha$

Incoming & reflected wave; near end of line, both ~ same amplitude, & give standing waves; but far away from end, one much larger than other; hence almost pure travelling wave (toward end). [at end, $|I|^2 \sim \sin^2$ - standing wave; at beginning, $|I|^2 \sim \sinh^2$; travelling wave with decreasing amplitude].

$I = R.P.:[2Ae^{i\omega t}(\sinh\beta(x_0-x)\cos\alpha(x_0-x) + i\cosh\beta(x_0-x)\sin\alpha(x_0-x))]$

Near end: $I = -2A\sin\alpha(x_0-x)\sin\omega t$. standing wave

Far from end: $I = 2A\cdot\frac{1}{2}e^{\beta(x_0-x)}\cos[\omega t+\alpha(x_0-x)]$ travelling wave

Intermediate case for $\beta(x_0-x)\sim 1$.

Potential: $u = Z\cdot 2A\cosh h(x-x_0)$

$= 2AZ[\cosh\beta(x_0-x_0)\cos\alpha(x_0-x_0) + i\sinh\beta(x_0-x)\sin\alpha(x$

Near end of line: $u = 2AZ\cos\alpha(x_0-x)$

Far from end: $\bar{u} = AZe^{\beta(x_0-x)}e^{i\alpha(x_0-x)}$.

$u = R.P.(\bar{u}e^{i\omega t}) = A|Z|e^{\beta(x_0-x)}\cos(\omega t+\alpha(x_0-x)+\varphi)$

where $Z = |Z|e^{i\varphi}$.

Near end: $u = 2|Z|A\cos\alpha(x_0-x)\cos(\omega t-\varphi)$

Phase shift $\pi/2+\varphi$ between I & u.

In this case, current node, potential loop at end.

Nov. 12, 1941

$$I = a e^{hx} + b e^{-hx}$$
$$U = -Zae^{hx} + Zbe^{-hx}$$
$$\frac{ae^{hx_0}}{be^{-hx_0}} = \frac{Z - R_e}{Z + R_e}$$

Case 2) : Short circuit : $R_e = 0$

then $ae^{hx_0} = be^{-hx_0} = A$

So $I = 2A \cosh h(x_0 - x)$

$U = 2AZ \sinh h(x_0 - x)$

i.e. potential node, current loop at end.

Same as previous case; standing waves near end, travelling waves near beginning of line.

Case 3) : $Z = R_e$;

then $a = 0$;

$$I = Ae^{h(x_0 - x)} = Ae^{i\alpha(x_0 - x) + \beta(x_0 - x)}$$
$$U = AZe^{h(x_0 - x)} = AZ \quad "$$
$$Ie^{i\omega t} = Ae^{i\alpha x_0} e^{\beta(x - x_0)} \underline{e^{i(\omega t - \alpha x)}}$$

Only travelling wave to end of line, & no reflected wave — this is important, because reflected wave is often undesirable, esp. in communication and in power transmission –

Krumhansl : Power line : Al. cable steel reinforced – 1" diam, 22 ft. apart; 220 kv @ 250 amp. at end; power factor .97 : 237 amp. at sending end.

If such $R_e = Z$, then takes 349 amp. at power station to give the same current at receiving end.

[$|Z + R_e| > |Z - R_e|$; because $Z = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$.

complex phase of numerator can vary between 0 & $\pi/2$;
same for denom; ∴ ratio has phase $< |\pi/2|$;
So if $Z = |Z| e^{i\varphi}$, $-\pi/4 < \varphi < \pi/4$.

Now $R_e = R + iL\omega + \frac{1}{iC\omega}$; phase again $< |\pi/2|$

So if $R_e = |R_e| e^{i\psi}$, $-\pi/2 < \psi < \pi/2$;

$$\therefore |Z + R_e|^2 = |Z|^2 + |R_e|^2 + Z R^* + Z^* R_e$$
$$|Z||R_e| e^{i(\varphi-\psi)} + |Z||R_e| e^{-i(\varphi-\psi)}$$
$$2 |Z||R_e| \cos(\varphi - \psi)$$

if $\varphi = \pi/4$, $\psi = -\pi/2$ then $\varphi - \psi = 3\pi/4$

$\cos(\varphi - \psi) = -\frac{1}{\sqrt{2}}$;

then $|Z - R_e|^2 > |Z + R_e|^2$! and
reflected wave larger than incident

Power transmission:

$$P = IU = Z\left(b^2 e^{-2hx} - a^2 e^{2hx}\right)$$

$$\frac{IU(\text{receiving end})}{IU(\text{sending end})} \simeq e^{-2hx_0}\left(1 - \left(\frac{a e^{hx_0}}{b e^{-hx_0}}\right)^2\right)$$

Nov. 14'41 [Problem.

Perturbation calculation of damping.

$$(L_1\alpha^2 + R_1\alpha + \tfrac{1}{C_1})(L_2\alpha^2 + R_2\alpha + \tfrac{1}{C_2}) = M^2\alpha^4$$

Neglecting R_1, R_2, get $-\alpha^2 = \left\{ \begin{matrix} \omega'^2 \\ \omega''^2 \end{matrix} ; \right\} = \frac{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4k^2\omega_1^2 \ldots}}{2(1-k^2)}$

∴ i.e. $L_1 L_2 (1 - k^2)(\alpha^2 + \omega'^2)(\alpha^2 + \omega''^2) = 0$.

Put in $R_1 R_2$; then $L_1 L_2 (1-k^2)(\alpha^2 + \omega'^2)(\alpha^2 + \omega''^2) = -R_1 \alpha L_2 (\alpha^2 + \omega_2^2)$
$- R_2 \alpha L_1 (\alpha^2 + \omega_1^2)$

let $\alpha = i\omega' + \beta$;

then $L_1 L_2 (1-k^2) \cdot \beta \cdot 2i\omega' (\omega''^2 - \omega'^2) = -2i[\omega' \beta_1 L_1 L_2 (\omega_2^2 - \omega'^2) + \omega' \beta_2 L_1 L_2 \ldots$

$\beta_1 = \frac{R_1}{2L_1}$ or $\beta = -\frac{\beta_1(\omega'^2 - \omega_2^2) + \beta_2(\omega'^2 - \omega_1^2)}{\ldots}$. $(\omega_1^2 - \omega'^2 \ldots$

for Tuned circuits:

$$-\alpha^2 = \frac{\omega^2(1 \pm k)}{1-k^2}$$

$$\omega'^2 - \omega^2 = \frac{\omega^2 k}{1-k} \qquad \omega'^2 = \frac{\omega^2}{1-k}$$

$$\omega'^2 - \omega''^2 = \frac{2\omega^2 k}{1-k^2} \qquad \omega''^2 = \frac{\omega^2}{1+k};$$

$$\therefore \beta' = -\frac{1}{2}\frac{\beta_1+\beta_2}{1-k} \leftarrow \text{large damping for high freq.}$$

$$\beta'' = -\frac{1}{2}\frac{\beta_1+\beta_2}{1+k} \leftarrow \text{small damping for low freq.}$$

Must see if β is negative:

$$\omega'^2 - \omega''^2 = \frac{\sqrt{}}{1-k^2}$$

$$\beta_1(\omega'^2-\omega_2^2) + \beta_2(\omega'^2-\omega_1^2) = (\beta_1+\beta_2)\left[\omega'^2 - \tfrac{1}{2}(\omega_1^2+\omega_2^2)\right] + (\beta_1-\beta_2)\left[\tfrac{1}{2}(\omega_1^2-\omega_2^2)\right]$$

it is!

Skin effect problem:

We had: $\frac{R_\omega}{R_0} = \frac{\lambda J_0(\lambda)}{2J_1(\lambda)}$, $\lambda = k r_0 = (1-i)k_0 r_0$.

To find setting in of skin effect

r_0 = radius of wire

$k_0 = \sqrt{\frac{2\pi\sigma\mu\omega}{10^9}}$.

Expand:

$$J_0 = 1 - \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4$$

$$J_1 = \frac{1}{2}\lambda\left(1 - \frac{1}{8}\lambda^2 + \frac{1}{192}\lambda^4\right).$$

$$\therefore \frac{\frac{\lambda}{2}J_0}{J_1} = ~~1-\left(\frac{1}{4}-\frac{1}{8}\right)~~ = 1 - \frac{1}{8}\lambda^2 - \frac{1}{192}\lambda^4$$

$$\lambda^2 = -2i k_0^2 r_0^2$$

$$\lambda^4 = -4k_0^4 r_0^4$$

$$\therefore \frac{R_\omega}{R_0} = 1 + \frac{1}{4}i(k_0 r_0)^2 + \frac{1}{48}(k_0 r_0)^4$$

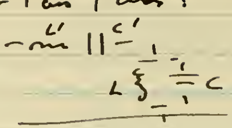
1st effect: increase of inductance

next; increase of resistance.

Resistance increase becomes appreciable @ $\sim k_0 r_0 \sim 2.5$

Band-Pass Filter:

Nov 17 '41



990 – 1010 kc.

$\omega_0 = 1,000$ kc

$LC = L'C' = \frac{1}{4\pi^2 \cdot 10^{12}} = 2.5 \cdot 10^{-14}$

Smallest cap. = 100 cm = $1.11 \cdot 10^{-10}$ F.

$$\frac{R_s}{2R_c} = -1$$

$$\therefore \; i\left(\omega L' - \frac{1}{\omega C'}\right) \cdot i\left(C\omega - \frac{1}{L\omega}\right) = -2$$

$$L'C\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 = 2$$

$$\therefore \; L'C = \frac{2}{\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}.$$

Since $\omega \approx \omega_0$,

$1 - \left(\frac{\omega_0}{\omega}\right)^2 \approx 2\left(1 - \frac{\omega_0}{\omega}\right)$ $\quad \therefore L'C \simeq \frac{1}{2\omega^2\left(1 - \frac{\omega_0}{\omega}\right)^2}$

$$= \frac{1}{2(\omega - \omega_0)^2}.$$

$$LC = \frac{1}{\omega_0^2}$$

$$\therefore \; \frac{L'}{L} = \frac{1}{2}\left(\frac{\omega_0^2}{(\omega - \omega_0)^2}\right).$$

$\omega_0 = 1000$, $\omega = 1{,}010$ $\quad \therefore \frac{L'}{L} = 5{,}000$.

$L' \gg L$, $C \gg C'$

$\therefore \; C' = 1.1 \cdot 10^{-10}$ F

$L' = 2.2 \cdot 10^{-4}$ H

$L = 4.4 \cdot 10^{-8}$ H

$C = 5.5 \cdot 10^{-7}$ F

At resonance,

$R_s = R'$

$$\frac{1}{R_c} = \frac{1}{Li\omega + R} + \frac{1}{i\omega C}$$

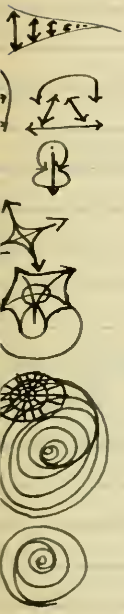
$$= -\frac{R}{(iL\omega)^2} = \frac{RC}{L}$$

$$\frac{R_s}{2R_c} = \sinh^2 \gamma/2 = \frac{R'RC}{2L} = \frac{1 \cdot .01 \cdot 5 \cdot 5 \cdot 10^{-7}}{2 \cdot 4.4 \cdot 10^{-8}} = .062$$

$\sinh \gamma/2 = .25$, $\quad e^{\gamma} = 1.6 \quad e^{-\gamma} = .6$

For 950 kc, $\sinh^2 \theta/2 \sim 25$, $\sinh \theta/2 \sim 5$

$\therefore \theta = 2.2$
$e^{2\theta} \sim 100$
$(10^5)^{th} = (10^2)^n$
$\therefore n = 2.5$ filters req'd.
to get 950 kc reduced by 10^5.



Transform:

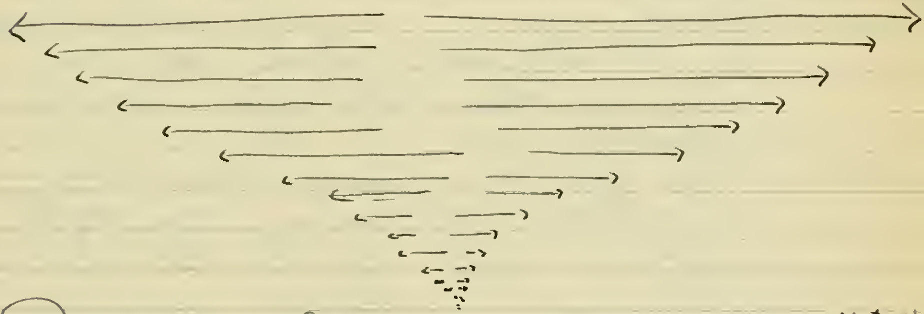
if $k = 1$,

get $\lambda_a = \sqrt{\frac{L_1}{L_2}}$; $\lambda_b = \sqrt{\frac{L_2}{L_1}}$.

$$\frac{U_2}{U_1} = \frac{I_1}{I_2} = \lambda$$

$\frac{U_2}{U_1} = R_a$ $\quad$ U_1 determined by λ_a

$\frac{U_2}{U_1} \simeq \lambda_a$ $\quad$ I_1 appreciable for both a & b.



Power at end of line: $\quad I(t)U(t) = I' e^{i\omega t} U' e^{i(\omega t - \varphi)}$

Time average $= \frac{1}{2} I' U'^*$

$U = R I$

$= \frac{1}{2} |I'|^2 R_E^*$

$= \frac{1}{2} |I'|^2 R$ (Real part)

$\{ = \frac{1}{2} |I'||U'| \frac{R}{|R|}$

$= \frac{1}{2} |I'||U'| \cos\varphi \}$.

Digression on complex products:

Power is really R.P.(I) · R.P.(U)

$= RP[I_r + iI_i][\cos\omega t + i \sin\omega t] \cdot R.P.[U_r + iU_i][\cos\omega t + i\sin\omega t]$

$= (I_r \cos\omega t - I_i \sin\omega t)(U_r \cos\omega t - U_i \sin\omega t)$

and $\overline{Pow} = \frac{1}{2}(I_r U_r + I_i U_i)$

$= \frac{1}{2}$ R.P. $I U^*$

Power along line:

$$RI^2 + GU^2.$$

$$= \left(-\frac{\partial U}{\partial x} - L\frac{\partial I}{\partial t}\right) I + \left(\frac{\partial I}{\partial x} - C\frac{\partial U}{\partial t}\right) U$$

$$= -\left\{ I\frac{\partial U}{\partial x} + U\frac{\partial I}{\partial x} + IL\frac{\partial I}{\partial t} + CU\frac{\partial U}{\partial t} \right\}$$

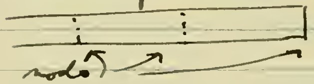
take average: of terms in $\frac{\partial I}{\partial t}$, $\frac{\partial U}{\partial t}$ vanish,

then $\overline{\text{Power}} = -\overline{\left(I\frac{\partial U}{\partial x} + U\frac{\partial I}{\partial x}\right)}$

$$= -\frac{\partial \overline{IU}}{\partial x}.$$

Nov. 19 '41 Standing waves: can get nodes & loops at any desired place by short circuiting at the right place; then U has node there, & other places: $U = A\sin\alpha(x_1 - x)$ if short circuit at x_1. $\alpha = \frac{\omega}{c'}$.

Can find nodes with neon tube; hence can measure wave length $\lambda = \frac{2\pi}{\alpha}$.



$$\lambda = \frac{2\pi}{\alpha} = \frac{2\pi c'}{\omega} = \frac{c'}{\nu} = \frac{c'}{c}\lambda_{vac}.$$

"Lecher wires"; used for obtaining dielectric constants at high freq; Put whole system in dielectric (e.g. a liquid), then $\alpha = \frac{\omega}{c'} = \omega\sqrt{LC}$

$C_\varepsilon = C_1\varepsilon$ C_ε = cap. in diel.
C_1 = " " air

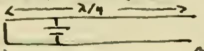
$$\therefore c'_\varepsilon = \frac{c'_1}{\sqrt{\varepsilon}}.$$

$$\therefore \lambda'_\varepsilon \sim \frac{1}{\sqrt{\varepsilon}}.$$

∴ Determine nodes in air, then in ε; ratio of λ's gives ratio of $\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$.

Large ~~potential~~ difference at ends (e.g. cyclotron — want large-as-possible pot.-diff. between dees).

To do this, try to make length of transmission line a quarter wave — (could use $3/4$ or any $\frac{2n+1}{4}$; but these are less stable —)



(voltage multiplication between excitation pt. & end.)

$u = 0$, $x = 0$ — excited here — $u = \max$

$$u = A \sin \alpha x \qquad \alpha l = \pi/2$$

$$l = \frac{\pi}{2\alpha} = \lambda/4$$

$$u_{end} = A\,; \qquad \lambda = \frac{2\pi}{\alpha} = 4l\,.$$

$$u_{exc.} = A \sin \alpha x_0 = A \sin \frac{\pi x_0}{2l}\,.$$

voltage multiplication by factor $\frac{1}{\sin \frac{\pi x_0}{2l}}$.

The working impedance at the end, of course modifies all this — (e.g. cyclotron dee capacity).

$$U = bZe^{-ikx} - aZe^{ikx}$$

$$\frac{ae^{il}}{be^{-il}} = \frac{Z - R_E}{Z + R_E} \qquad \text{max. of } u \text{ at end, only for } R_E \to \infty.$$

$R_E \gg Z$ $\qquad R_E = \frac{1}{iC\omega} = \frac{1}{C\omega} e^{-i\pi/2}$ $\qquad Z = \sqrt{\frac{L'}{C'}}$ $\quad L', C'$ per unit length

define : $\tan\gamma = \frac{R_E}{iZ}$

$$\frac{ae^{kl}}{be^{-kl}} = Ae^{2i\gamma}, \quad ae^{kl} = Ae^{i\gamma}, \quad be^{-kl} = Ae^{-i\gamma}$$

then $U = 2ZAi$ ~~sin(αx−γ)~~ $\sin[\alpha(l-x)+\gamma]$.

Resonance if $u = 0$ at $x = 0$

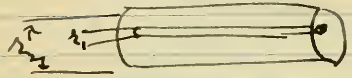
$$\therefore \alpha l + \gamma = \pi/2 \qquad \therefore l = \frac{\pi/2 - \gamma}{\alpha}\,.$$

$$\frac{2\pi l}{\lambda'} = \pi - \gamma$$

$$l = \lambda'\left(\frac{1}{2} - \frac{\gamma}{2\pi}\right)$$ (if $R_c = \infty$, $\frac{\pi}{2}$ $\gamma = \frac{\pi}{2}$ then $l = \lambda'/4$ as before)

But since actually γ is between $\frac{\pi}{2}$ + π, length is shorter : $l < \frac{1}{4}\lambda'$.

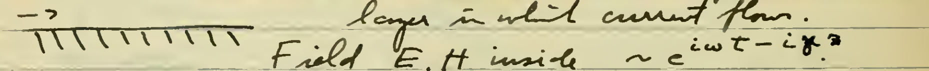
Problem: Transmission line of concentric conductors :



Assume $C_{dees} = 100$ cm.

take $1 > \frac{r_1}{r_2} > \frac{1}{e}$

30 meter waves.



Nov. 24 Field in hollow conductors

cylinder — Thick walls: thick compared with layer in which current flows.

Field E, H inside $\sim e^{i\omega t - i\gamma z}$

take coordinate z along cylinder. Interested in solution with $\gamma = \alpha - i\beta$, $\alpha \gg \beta$ (small damping). Phase velocity $\frac{\omega}{\alpha}$ · may be greater than c ; (group velocity will never be).

In the vacuum : $\frac{1}{c}\dot{\underline{H}} = -\nabla \times \underline{E}$. $\nabla \cdot \underline{E} = 0$

$\frac{1}{c}\dot{\underline{E}} = \nabla \times \underline{H}$ $\nabla \cdot \underline{H} = 0$

In conductor : $\frac{1}{c}\dot{\underline{E}} + \frac{4\pi}{c}\underline{j} = \nabla \times \underline{H}$

neglect ↗

At surface of conductor, tangential comp of E + H must be continuous

We expect tangential component of E to be very small, so as to avoid large losses. Hence for 1st approximation, say it vanishes. : $E_z = 0$, $E_\varphi = 0$ at surface of conductors

No condition on E_r or $\underline{H}$.

> Carson
> Mead
> Schelkunoff
> Bell. Tel. Jour.
> vol 15, p. 310 '36

$$(\nabla \times \underline{E})_z = \frac{1}{r}\frac{\partial}{\partial r}(r E_\varphi) - \frac{1}{r}\frac{\partial}{\partial \varphi} E_r$$

$$(\nabla \times E)_r = \frac{1}{r}\frac{\partial}{\partial \varphi} E_z - \frac{\partial E_\varphi}{\partial z}$$

$$(\nabla \times E)_\varphi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}$$

Now $\frac{1}{c}\dot{H}_\varphi = -\frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r}$.

Put in time & z-dependence:

$$i\frac{\omega}{c} H_\varphi = i\gamma E_r + \frac{\partial E_z}{\partial r}$$

Let $k = \frac{\omega}{c}$.

$$H_\varphi = \frac{\gamma}{k} E_r - \frac{i}{k}\frac{\partial E_z}{\partial r}$$

likewise $\frac{1}{c}\dot{E}_r = \frac{1}{r}\frac{\partial}{\partial \varphi} H_z - \frac{\partial H_\varphi}{\partial z}$

or $-ik E_r = \frac{1}{r}\frac{\partial H_z}{\partial \varphi} + i\gamma H_\varphi$

$$E_r = -\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k} H_\varphi .$$

$$\therefore H_\varphi = \frac{\gamma}{k}\left[-\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k} H_\varphi\right] - \frac{i}{k}\frac{\partial E_z}{\partial r}.$$

$$\therefore H_\varphi\left(1 - \frac{\gamma^2}{k^2}\right) = -\frac{i}{k}\left[\frac{\gamma}{k}\frac{1}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\partial E_z}{\partial r}\right]$$

Let $\lambda^2 = k^2 - \gamma^2$

$$\lambda^2 H_\varphi = -i\left[k\frac{\partial E_z}{\partial r} + \frac{1}{r}\gamma\frac{\partial H_z}{\partial \varphi}\right].$$

Can also ~~[illegible]~~ H_φ: $E_r = -\frac{i}{rk}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k}\left[\frac{\gamma}{k} E_r - \frac{i}{k}\frac{\partial E_z}{\partial r}\right]$

$$\therefore E_r\left(1 - \frac{\gamma^2}{k^2}\right) = -\frac{i}{k}\left[\frac{1}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\gamma}{k}\frac{\partial E_z}{\partial r}\right]$$

$$\therefore \lambda^2 E_r = -i\left[\frac{k}{r}\frac{\partial H_z}{\partial \varphi} + \gamma\frac{\partial E_z}{\partial r}\right]$$

Can get similar eq'ns with E, H interchanged:

$$\lambda^2 E_\varphi = -i\gamma/r \frac{\partial E_z}{\partial \varphi} + ik \frac{\partial H_z}{\partial r}.$$

and $$\lambda^2 H_r = -i\gamma \frac{\partial H_z}{\partial r} + \frac{ik}{r}\frac{\partial E_z}{\partial \varphi}.$$

So we have all r & φ components expressed in terms of z-comp.

To get an eq. for z-comp., use divergence condition:

$$\nabla \cdot \underline{E} = 0$$

i.e. $$\frac{\partial E_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{1}{r}\frac{\partial E_\varphi}{\partial \varphi} = 0$$

$$-i\gamma \lambda^2 E_z + \frac{1}{r}\frac{\partial}{\partial r}\left(-ik\frac{\partial H_z}{\partial \varphi} - i\gamma r \frac{\partial E_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left[-\frac{i\gamma}{r}\frac{\partial E_z}{\partial \varphi} + ik\frac{\partial H_z}{\partial r}\right] = 0$$

or: $$-i\gamma\lambda^2 E_z - ik \cdot \frac{1}{r}\frac{\partial^2 H_z}{\partial r \partial \varphi} - i\gamma \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - i\gamma \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \frac{ik}{r}\frac{\partial}{\partial \varphi}\frac{\partial H_z}{\partial r} = 0$$

(cancel)

$$\therefore \quad i\gamma\lambda^2 E_z - i\gamma \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - i\gamma \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} = 0$$

or: $$\lambda^2 E_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} = 0$$

This is just wave eq'n, which would have come from $\nabla^2 \underline{E} = -k^2 \underline{E}$

Nov. 26, '41

$$\begin{cases} \nabla^2 E_z + k^2 E_z = 0 \\ \nabla^2 H_z + k^2 H_z = 0 \end{cases}$$

($\nabla^2 E_\varphi + k^2 \varphi = 0$ wrong!
$(\nabla^2 + k^2)$ ~ holds only for Cartesian coord's.

[because div grad $\underline{A} = \nabla^2 \underline{A}$ only in cartesian coords.

We had $\lambda^2 = k^2 - \gamma^2$: (γ gives z-dependence)

then $$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \lambda^2 E_z = 0$$

this gives $E_z = A\, J_m(\lambda r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi$

m an integer ≥ 0.

J_m must be an ordinary Bessel f'n, not a Hankel or Neumann function, since field must remain finite at $r=0$.

Also: $H_z = B\, J_m(\lambda r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi$.

Boundary conditions: $r = R$: (radius of tube)

$E_z = 0$, $E_\varphi = 0$ $\quad (E_{\parallel} = 0)$

We had: $$\lambda^2 E_\varphi = -i\,\gamma/r \frac{\partial E_z}{\partial \varphi} + i k \frac{\partial H_z}{\partial r}$$
$$= -\frac{i\gamma}{r} A\, J_m(\lambda r)\, m \begin{smallmatrix}-\sin\\ \cos\end{smallmatrix} m\varphi + ik B\, J_m'(\lambda r)\cdot\lambda \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi$$

To make $E_z = 0$, $r = R$: either $A = 0$

or $J_m(\lambda R) = 0$

in either case, 1st term of E_φ vanishes

To make 2nd " " " " : either $B = 0$

or $J_m'(\lambda R) = 0$

Now $J_m(\lambda R) = 0$ and $J_m'(\lambda R) = 0$ cannot be simultaneously true (roots of Bessel f'n + derivatives do not come at same place).

$\therefore$ if ~~...~~ $J_m(\lambda R) = 0$, then $B = 0$

if $J_m'(\lambda R) = 0$, then $A = 0$

$\therefore$ 2 classes of solutions. 1) $H_z = 0$ identically "E sol'ns"

2) $E_z = 0$ " "H sol'ns"

{ For large x, $J_n(x) \sim \sin(x - (n-\frac{1}{2})\frac{\pi}{2})$

$J_n'(x) \sim \sin(x - (n-\frac{3}{2})\frac{\pi}{2})$

Zeros of J_n: $x = m\pi + (n-\frac{1}{2})\frac{\pi}{2}$

$= (m + \frac{n}{2})\pi - \frac{\pi}{4}$

all even n have same roots, odd n same roots.

Zeros of J_n' same as $J_{n\pm 1}$. } .

Since we are interested in modes with only a few nodes in radial direction, above doesn't apply. i.e. Important solutions are for small n & x.

Roots : E solutions

$m = 0$ (E, H, indep. of φ).

$J_0(\lambda R) = 0$ at $\lambda R = 2.4048$ (Jahnke - Emde)

5.5201

⋮

Call $(R\lambda_{m\mu})$ the μth root of the mth Bessel f'n. i.e. $J_m(\lambda_{m\mu} R) = 0$

$m = 1$

$J_1(\lambda R) = 0$: 3.8317

7.0156

H solutions

n ~~m~~ $= 0$

$J_0'(\lambda R) = 0$ ($J_0' = -J_1$)

∴ 3.8317 . (so in this case can have

7.0156 E & H sol'ns, at once)

$n = 1$: $J_1'(\lambda R) = 0$ at $\lambda R = 1.84$

5.33

Each solution gives a characteristic field distribution which is easily calculated (pictures in Southworth, Bell Journal)

$$\lambda^2 E_r = -i\gamma \frac{\partial E_z}{\partial r} - \frac{ik}{r}\frac{\partial H_z}{\partial \varphi}$$

E_r is out of phase with E_z, or H_z

similarly H_r, H_φ, E_φ ; but are in phase with each other.

Possible values of k : $k^2 = \gamma^2 + \lambda^2$; γ almost real.

discrete set of possible λ's ; $\lambda = \frac{x_{m\mu}}{R}$.

smallest $\lambda = \frac{1.84}{R}$ (H_{11} mode). | $J_m(x_{m\mu}) = 0$

$\therefore k > \lambda_{m\mu} = \frac{1.84}{R}$ at least.

$\therefore$ only for suff. great frequencies can these be excited.

$k = \frac{2\pi}{\lambda_{vac}}$ $\quad \lambda_{vac} = \frac{2\pi}{k} < \frac{2\pi}{\lambda_{m\mu}} = \frac{2\pi R}{x_{m\mu}}$.

biggest possible wave length $\sim \frac{2\pi}{1.84} R \sim 3.3 R$

no greater wave length can exist in these tubes.

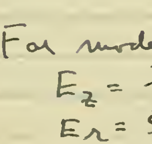
Nov. 28 '41 For mode E_{01} :

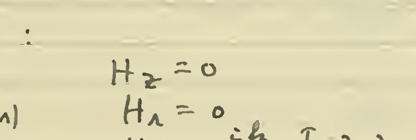
$E_z = J_0(\lambda r)$ $\qquad H_z = 0$

$E_r = \frac{i\gamma}{\lambda} J_1(\lambda r)$ $\qquad H_r = 0$

$E_\varphi = 0$ $\qquad H_\varphi = \frac{ik}{\lambda} J_1(\lambda r)$.

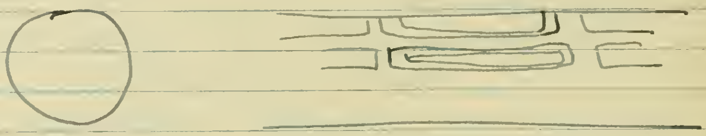
} factor $e^{-i\gamma z}$ throughout





H_{01} : same as previous except E & H interchanged — except that condition is now $E_\varphi = \frac{ik}{\lambda} J_1'(\lambda r) = 0$ for $r = R$ (at wall) ∴ $H_r = 0$ at wall, H_z a max. at wall;

E_{02} :



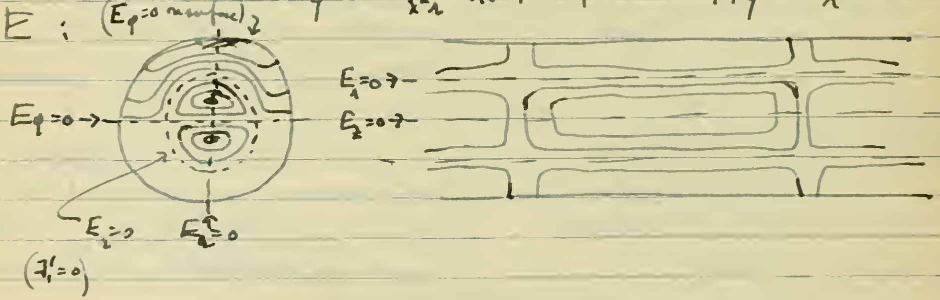
E_{11} :

$$E_z = J_1(\lambda r)\cos\varphi \qquad H_z = 0$$

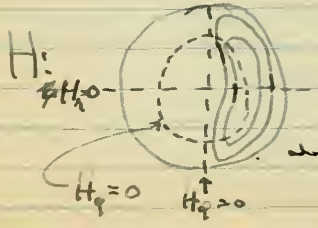
$$E_r = -\frac{i\gamma}{\lambda} J_1'(\lambda r)\cos\varphi \qquad H_r = \frac{ik}{\lambda^2 r} J_1 \sin\varphi$$

$$E_\varphi = -\frac{i\gamma}{\lambda^2 r} J_1(\lambda r)\sin\varphi \qquad H_\varphi = -\frac{ik}{\lambda} J_1' \cos\varphi$$

E :

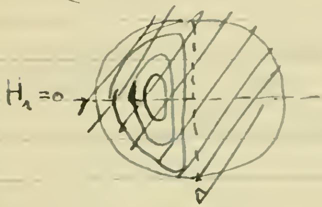


H:

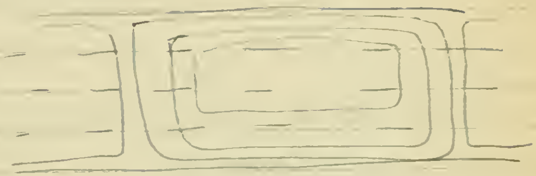


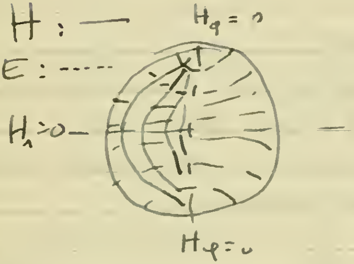
H_{11} (lowest possible λ : $\lambda R = 1.8$).

($J_1' = 0$ at wall).



H lines :





(Southworth).

E_{01} wave looks something like putting a current going down axis — inside a shield.

E_{11} is like two currents going down pipe —

Dec. '41 E-wave:

$$\lambda^2 E_r = -i\gamma \frac{\partial E_z}{\partial r}$$
$$\lambda^2 E_\varphi = -\frac{i\gamma}{r}\frac{\partial E_z}{\partial \varphi}$$
$$\lambda^2 H_r = -\frac{ik}{r}\frac{\partial E_z}{\partial \varphi}$$
$$\lambda^2 H_\varphi = ik\frac{\partial E_z}{\partial r}.$$

$$\left\{ \begin{array}{l} \lambda^2 = k^2 - \gamma^2. \\ J_n(\lambda R) = 0 \end{array} \right.$$

$$k = \frac{\omega}{c} = \sqrt{\lambda^2 + \gamma^2}.$$

For infinitely long line, any value of γ is allowed; otherwise depend on way the line is closed.

Suppose closed at both ends :

at ends $E_r = E_\varphi = 0$.

$E_r, E_\varphi \sim \sin \gamma z$

$\therefore \gamma L = K\pi$ K integer

$\gamma = \frac{K\pi}{L}$. discrete set of γ's

but if $L \to \infty$, become practically continuous.

Infinite line:

may define phase velocity:

$e^{i(\omega t - \gamma z)}$

$$v_p = \frac{\omega}{\gamma} = \frac{ck}{\gamma} = c\,\frac{\sqrt{\gamma^2+\lambda^2}}{\gamma} > c .$$

$v_p \to \infty$ for $\gamma \to 0$

$v_p \to c$ for $\gamma \gg \lambda$.

But group velocity must not exceed c:

$v_g \neq v_p$ if v_p depends on ω.

$$v_g = \frac{d\omega}{d\gamma} . \qquad \omega = c\sqrt{\lambda^2+\gamma^2}$$

$$= c \cdot \frac{\gamma}{\sqrt{\gamma^2+\lambda^2}} < c .$$

$v_p v_g = c^2$

→ consider a wave packet:

$\psi = \int d\gamma\, a(\gamma)\, e^{i(\omega t - \gamma z)}$

(only $\gamma \approx \gamma_0$ occur) choose $a(\gamma)$ so that ψ is large only for $z \approx 0$ at $t=0$; ~~at a time t the maximum~~

$2\Delta\gamma \gg 1$

then $\psi \approx e^{i(\omega_0 t - \gamma_0 z)}$

$\int d\gamma\, a(\gamma)\, e^{i(\gamma-\gamma_0)(\frac{d\omega}{d\gamma} t - z)} + \dots$

$= f\left[\frac{d\omega}{d\gamma} t - z\right]$

Hence $v_g = \frac{d\omega}{d\gamma}$ in 1st approx.

If $E_z = J_n(\lambda r)\cos n\varphi$,

$E_\varphi = -i\,\frac{\gamma}{\lambda}\,\frac{J_n(\lambda r)}{\lambda r}\, n \sin n\varphi$

for low modes,

$\frac{E_\varphi}{E_z} \sim \gamma/\lambda$.

$\frac{E_r}{E_z} \sim \gamma/\lambda$

similarly H_φ, H_r;

$\therefore \frac{\text{Transverse comp.}}{\text{Longitudinal comp.}} \sim \gamma/\lambda$.

if $\lambda \sim \gamma$, all comp. are of same order of mag.;

But if $\gamma \gg \lambda$ Transverse comp. $\gg$ long. comp.

Then get ^very nearly^ plane wave in z-direction with velocity very nearly vel. of light. — this case represents transition from line-waves to free space-waves.

Energy propagation along line.

Can be done two ways — (1) calculate energy content per unit ~~volume~~ ^length^, & multiply by group velocity; gives total energy transmitted thru cross-section of tube per sec.
(2) calculate Poynting vector.

(1) Consider E-wave;

want $\frac{1}{8\pi}(E^2+H^2)$ for a unit length:

$$\overline{W} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^R (E^2+H^2)\, r\, dr\, d\varphi$$

(don't forget we have been using complex representation)

$E = E' e^{i\omega t}$

$H = H' e^{i\omega t}$

let $x = \lambda r$

$$\overline{W} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^R \frac{1}{2}\left((E')^2+(H')^2\right) r\, dr\, d\varphi$$

$$= \frac{1}{16\pi}\int_0^{2\pi}\int_0^R \Big[\Big(J_n^2\cos^2 n\varphi + \frac{\gamma^2}{\lambda^2}\cdot\frac{n^2}{x^2}J_n^2\sin^2 n\varphi + \frac{\gamma^2}{\lambda^2}J_n'^2(x)\cos^2 n\varphi\Big) + \Big(\frac{k^2}{\lambda^2}\cdot\frac{n^2}{x^2}J_n^2\sin^2 n\varphi + \frac{k^2}{\lambda^2}J_n'^2(x)\cos^2 n\varphi\Big)\Big]\frac{x\,dx\,d\varphi}{\lambda^2}$$

$$= \frac{1}{16\pi}\cdot\frac{\pi}{\lambda^2}\int_0^R x\,dx\left[J_n^2 + \frac{\gamma^2+k^2}{\lambda^2}\left(J_n'^2 + \frac{J_n^2 n^2}{x^2}\right)\right].$$

(2) Poynting vector: $S_z = \frac{c}{4\pi}(\underline{E}\times\underline{H})_z$ (per sq. cm)

$$= \frac{c}{4\pi}\left[E_r H_\varphi - E_\varphi H_r\right]$$

total transf. $= R.P.\ \frac{c}{4\pi}\cdot\frac{1}{2}\int_0^{2\pi}\int_0^R (E_r H_\varphi^* - E_\varphi H_r^*)\, r\, dr\, d\varphi$.

$\therefore$ ~~total~~ transfer $= \frac{c}{8\pi} \int \left(\frac{\gamma h}{\lambda^2} \left(\frac{\partial E_z}{\partial x} \right)^2 + \frac{\gamma h}{\lambda^2} \frac{1}{x^2} \left(\frac{\partial E_z}{\partial \varphi} \right)^2 \right) \frac{x\,dx\,d\varphi}{\lambda^2}$

$= \frac{c}{8\pi} \cdot \frac{\gamma h}{\lambda^2} \int \left(J_n'^2 \cos^2 n\varphi + \frac{1}{x^2} J_n^2 n^2 \sin^2 n\varphi \right) \frac{x\,dx\,d\varphi}{\lambda^2}$

$= \frac{c}{8\pi} \cdot \pi \cdot \frac{\gamma h}{\lambda^4} \int \left(J_n'^2 + \frac{n^2}{x^2} J_n^2 \right) x\,dx$.

$\int J_n'^2 x\,dx = J_n \cdot x J_n' \Big|_0^{\lambda R} - \int_0^{\lambda R} J_n \cdot \frac{d}{dx}(x J_n')\, x\,dx$

at λR, $J_n = 0$ for E wave, $J_n' = 0$ for H wave.

$\therefore \int J_n'^2 x\,dx = \int_0^{\lambda R} J_n [$

$\langle + \frac{n^2}{x^2}$

anyhow, $\int \left(J_n'^2 + \frac{n^2}{x^2} J_n^2 \right) x\,dx = \int J_n^2 x\,dx$

so by (1), get: $\overline{W} = \frac{1}{16\lambda^2} \int x\,dx \left\{ 1 + \frac{\gamma^2 + h^2}{\lambda^2} \right\}$

$= \frac{1}{16\lambda^2} \cdot \frac{2h^2}{\lambda^2} \int J_n^2 x\,dx$.

$\overline{W} \cdot v_g$

$= \frac{h^2}{8\lambda^4} \cdot c \frac{\gamma}{h} \cdot \int J_n^2 x\,dx$

$= \frac{\gamma h}{8\lambda^4} \int J_n^2 x\,dx$

& this is same as result by (2) .

Dec 3, '41 Bethe forgot to set alarm.

Dec. 5, '41 We found Energy transport $S = Wc_g = W\frac{c\gamma}{k}$ W = energy/cm in tube.

$$S = \frac{c\gamma k}{8\lambda^4}\int_0^{x_0} J_n^2\, x\, dx. \qquad x_0 = \lambda R.$$

$J_n(x_0) = 0$ for E-wave

$J_n'(x_0) = 0$ for H-wave.

Now $$\int_0^{x_0} J_n^2 x\,dx = \tfrac{1}{2}x^2\left(J_n^2 - J_{n-1}J_{n+1}\right)\Big|_0^{x_0}$$

but $J_{n-1} = J_n' + \frac{n}{x}J_n$

$J_{n+1} = -J_n' + \frac{n}{x}J_n$

$$\therefore \int_0^{x_0} J_n^2\, x\, dx = \tfrac{1}{2}(x^2 - n^2)J_n^2 + \tfrac{1}{2}x^2 J_n'^2\Big|_0^{x_0}$$

$$= \tfrac{1}{2}(x_0^2 - n^2)J_n^2(x_0) + \tfrac{1}{2}x_0^2 J_n'^2(x_0)$$

for E-wave: $\tfrac{1}{2}x_0^2 J_n'^2(x_0)$

H-wave: $\tfrac{1}{2}(x_0^2 - n^2)J_n^2(x_0)$

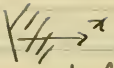
Energy consumed in wall per sec per cm of length: P

P/W = fraction of energy consumed per cm.

So that $E_z = E_{z_0}e^{-\beta z}$ $\beta = \frac{P}{2W}$.

(this valid because β will turn out to be $\ll \lambda$, k, etc.

Energy consumption due to currents in surface of tube — We had: $E_{tang} = 0$ at wall; but in general $H_{tang} \neq 0$; But H must vanish soon (beyond R) hence H decreases (exponentially) as we go into metal; there is a radial current $j = \sigma E$ on acc't of radial E; this causes a curl H, so that H(r). Current exists in a very small sheath in comp. with size of tube; hence don't have to use Bessel f'ns — tube wall may be considered plane.

If τ is cond. into metal :  then is a current $j(x)$; and a Joule heat $\frac{1}{\sigma}\int j^2(x)\,dx \; dS$ (dS - surface element of tube) . also have a mag. field $H_{tan}(x)$. Call y ~~the direction~~ of H .

$$j = \nabla\times H = \frac{\partial H_y}{\partial x} \qquad \therefore j \text{ in } z\text{-direction}$$

(not previous z - ⊥ to present x & y)

In metal : $\nabla^2 j = \frac{4\pi i\sigma\mu\omega}{c^2}\, j$

essentially :

$\left(\frac{\sigma_{esu}}{c^2} = \sigma_{emu} = \sigma_{mho}\cdot 10^{-9}\right)$

$$\frac{\partial^2 j}{\partial x} = (\;)\, j$$

$$\therefore \quad j = B\, e^{\pm\sqrt{4\pi i\sigma\mu\omega}\; x}$$

$$= B\, e^{\pm\sqrt{2\pi\sigma\mu\omega}\,(1+i)\,x} \qquad \text{throw out + sign}$$

let $\frac{1}{D} = \sqrt{2\pi\sigma\mu\omega}$

[D ~ thickness of current bearing sheath] so that $\int j^2 dx = $ ~~$B^2 \int e^{-2(1+i)x/D}$~~

$$= B^2 \int e^{-\frac{2x}{D}} = \frac{1}{2} B^2 D .$$

still need $B := j(0)$.

$$\nabla\times H = \frac{4\pi}{c}\, j = \frac{\partial H_y}{\partial x}$$

$$\frac{4\pi}{c}\int j\,dx = (H_y)_{x=0} - (H_y)_{\text{inside metal}} \qquad (=0)$$

$$\therefore (H_y)_{x=0} = \frac{4\pi}{c}\, B D .$$

$$\therefore \text{Joule heat} = \frac{1}{\sigma}\, dS \cdot \frac{1}{2} D \cdot \left(\frac{c}{4\pi}\right)^2 \cdot \frac{1}{D^2}\,(H_{tan})^2$$

$$\text{total} \;\ldots\; = \frac{c^2}{32\pi^2\sigma D}\int H_{tan}^2\, dS \quad \text{over surface of tube.}$$

$$= \frac{1}{32\pi^2\sigma_{emu} D}\int H^2\, R\, d\varphi\, dz$$

Power consumption per cm :

$$P = \frac{R}{32\pi^2\sigma D}\int H_{tan}^2\, d\varphi$$

since $D \sim \frac{1}{\sqrt{\sigma}}$, $P \sim \frac{1}{\sqrt{\sigma}}$ ∴ small loss for good conductor.

Now: $H_{tan}^2 = H_z^2 + H_\varphi^2$ E wave: $H_z = 0$, $H_\varphi \neq 0$

H-wave: ~~both~~ $H_z \neq 0$,

$H_\varphi \sim \frac{\partial H_z}{\partial \varphi} \begin{cases} = 0, & n = 0 \\ \neq 0, & n \neq 0 \end{cases}$.

Now $\frac{\text{trans}}{\text{long.}} \sim \frac{h}{\lambda}$; ∴ tran. comp. increase with freq. relative to longitudinal; Hence at high freq., H_z is small, H_φ large; Hence the H_0 wave will be the best at high freq; smallest attenuation, since only $H_z \neq 0$ ~~at~~ for $n = 0$.

[But unstable for such high freq anyhow].

usually $h \approx \lambda$ – there it doesn't matter.

Dec. 8, '41 We had: Power consumption per cm.:

$$P = \frac{R}{32\pi^2 \sigma D} \int H_{tan}^2 \, d\varphi$$

E waves: $H_{tan} = H_\varphi = \frac{i l}{\lambda^2} \frac{\partial E_z}{\partial r}$.

$$E_z = J_n(\lambda r) \cos n\varphi$$

$$\frac{\partial E_z}{\partial r} = \lambda J_n'(\lambda r) \cos n\varphi$$

$$\therefore \int |H_{tan}|^2 d\varphi = + \frac{l^2}{\lambda^2} J_n'^2(x_0) \cdot \pi$$

$$\therefore P = \frac{1}{32\pi} \frac{R}{\sigma D} \frac{l^2}{\lambda^2} J_n'^2(x_0) \qquad x_0 = \lambda r$$

H waves: $H_{tan}^2 = H_\varphi^2 + H_z^2$

$$H_z = J_n \cos n\varphi$$

$$H_\varphi = -\frac{i\gamma}{\lambda^2} \frac{1}{r} \frac{\partial H_z}{\partial \varphi} = +\frac{i\gamma}{\lambda^2 r} J_n \, n \sin n\varphi$$

$$H_{tan}^2 = J_n^2 \left[\cos^2 n\varphi + \frac{\gamma^2 n^2}{\lambda^4 r^2} \sin^2 n\varphi\right]$$

$$\therefore \int H_{tan}^2 \, d\varphi = J_n^2 \left[1 + \frac{\gamma^2 n^2}{\lambda^2 x_0^2}\right] \pi$$

$$\therefore \; P = \frac{1}{32\pi}\,\frac{R}{\sigma D}\,J_n^2(x_0)\left(1 + \frac{h^2}{\lambda^2}\,\frac{n^2}{x_0^2}\right).$$

$P \sim \omega^2$ if λ is fixed + ω varied for E-wave;
For H waves: for $n=0$; second term missing; smaller power consumption than for any other wave.

We had before: Energy transmitted per sec:

$$S = \frac{c\gamma h}{8\lambda^4}\cdot\frac{1}{2}\left[J_n^2(x_0^2-n^2) + x_0^2 J_n'^2\right].$$

E waves: $J_n(x_0)=0$

$$P/S = \frac{1}{2\pi}\cdot\frac{R}{\sigma D}\cdot\frac{h}{c\gamma}\cdot\frac{\lambda^2}{x_0^2} \qquad = \lambda^2 R^2$$

$$= \frac{h}{\gamma}\,\frac{1}{c\,2\pi\,\sigma D R}.$$

but $D = \frac{1}{\sqrt{2\pi\sigma\mu\omega}}$

$$\therefore \sigma = \frac{1}{2\pi\mu\omega D^2}$$

$$\therefore P/S = \frac{h}{\gamma c}\,\frac{\mu\omega D}{R}.$$

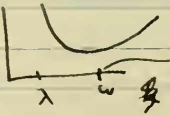
but $\omega/c = h^2$: $\quad P/S = \frac{\mu h^2}{\gamma}\,\frac{D}{R}$

Don't use iron; makes large P/S

attenuation $\beta = \frac{P}{2S} \sim \omega^{1/2}$ at high freq.

(actually $\beta \sim \frac{\omega^{3/2}}{\sqrt{\omega^2-\omega_0^2}} = f \qquad \omega_0 = c\lambda$ for all ω

$$\frac{\partial f}{\partial\omega} = \frac{3/2\,\omega^{1/2}}{\sqrt{}} - \frac{\omega^{3/2}\cdot\omega}{(\sqrt{})^3} = 0$$



$$\frac{3}{2}(\omega^2-\omega_0^2) = \omega^2$$

$$\therefore 3\omega^2 - 2\omega^2 = 3\omega_0^2$$

$$\omega = \sqrt{3}\,\omega_0$$

We found for E_0 wave, smallest λ : $\lambda R = 2.4$

$\therefore k_{min} = \frac{2.4}{R} \cdot \sqrt{3}$ and wave length $\Lambda_{min} = \frac{2\pi}{k_{min}} = \frac{2\pi}{2.4\sqrt{3}} R$.

$\sim 1.4_5 R$.

also $\beta \sim \frac{D}{R}$ which is small. $<< 1$.

we found before : $D = \frac{1}{\sqrt{2\pi\sigma\mu\omega}}$ take $\mu = 1$

$= \frac{1}{2\pi\sqrt{\sigma f}}$ $f = 2\pi\omega =$ freq.

for Cu : $D = \frac{1}{2\pi\sqrt{f}\sqrt{6 \cdot 10^{-4}}}$

$= \frac{1}{.15\sqrt{f}}$

if $f \sim 10^{10}$, (3 cm waves)

$D = \frac{1}{.15 \cdot 10^5} \sim 7 \cdot 10^{-5}$ cm.

For best operation : $R \sim 2/3\, \Lambda$

$\therefore R \sim 2$ cm.

and $D/R \sim 3.5 \cdot 10^{-5}$ cm.

and $\beta \sim 2 \cdot 10^{-5}$ cm.

Very small attenuation; doesn't affect field very much at any one place. This justifies considering $\gamma + \lambda^2$ real; actually $\gamma = \alpha + i\beta$, $\lambda^2 = k^2 - \gamma^2 = k^2 - \alpha^2 - 2i\alpha\beta$, but these corrections are certainly negligible.

$\beta \sim 2$ per km. about the same as what we found for ordinary lines at moderate freq. (~1000

$\beta_{min} = \sqrt{\frac{2}{3}}\, \frac{2\pi}{\Lambda}\, \frac{1}{.15\sqrt{f}} \cdot \frac{1}{R}$ for Cu

$R \sim \Lambda \sim \frac{1}{f}$

$\therefore \beta_{min} \sim f^{3/2}$ for given mode.

This gets worse for high freq.

H-waves: $\left[J_n'(x_0)=0\right]$

$$P/S = \frac{1}{2\pi c}\cdot\frac{R}{\sigma D}\cdot\frac{\lambda^4}{\gamma k}\cdot\frac{1+\frac{n^2\gamma^2}{x_0^2\lambda^2}}{x_0^2-n^2} \qquad x_0=\lambda R$$

$$= \frac{1}{2\pi c}\cdot\frac{1}{\sigma D}\cdot\frac{\lambda^2}{\gamma k}\,\frac{x_0^2+\gamma^2/\lambda^2\, n^2}{(x_0^2-n^2)R}$$

$$= \frac{1}{2\pi c}\cdot\frac{1}{\frac{1}{2\pi\sigma\mu D^2}\cdot D}\,\frac{1}{\gamma k}\,\frac{\lambda^2 x_0^2+\gamma^2 n^2}{(x_0^2-n^2)R}$$ [illegible]

$$= \frac{\sigma\mu}{c}\cdot\frac{D}{R}\cdot\frac{1}{\gamma k}\,\frac{\lambda^2 x_0^2+\gamma^2 n^2}{(x_0^2-n^2)}$$

for $n\neq 0$, $\gamma\gg$ [illegible]: $\mu k\frac{D}{R}\cdot\frac{\gamma}{k}\,\frac{n^2}{x_0^2-n^2}$.

$n=0$: $\mu k\frac{D}{R}\cdot\frac{\lambda^2}{\gamma k}$.

for small γ k, same as E-wave. (except $n=0$)

Dec. 10, '41 We found:

E wave, $n\neq 0 \to$ H wave: $\beta \sim \frac{\omega^{3/2}}{\sqrt{\omega^2-\omega_0^2}}$

H_0 wave: $\beta \sim \frac{1}{\sqrt{\omega}\sqrt{\omega^2-\omega_0^2}}$

attenuation.

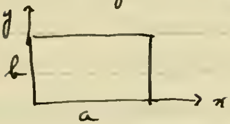
Elliptical cross-section: Chu, J. App. Phys. 9, 583.

Two different types of waves: $\cos n\varphi$ & $\sin n\varphi$ no longer degenerat

λ is determined essentially by major axis in one case by minor axis in the other.

Waves even with resp. to maj axis have higher λ than odd waves.

Rectangular cross-section. (Barrow - J. Appl. Phys.



Maxwell's eq'n's:

$$\frac{i\omega}{c} E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

$$-ih\, H_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}$$

$$(h^2-\gamma^2)\, E_x = -i\gamma \frac{\partial E_z}{\partial x} - ih \frac{\partial H_z}{\partial y}$$

$$\lambda^2 H_y = -i\gamma \frac{\partial H_z}{\partial y} - ih \frac{\partial E_z}{\partial x}.$$

All quantities ~ $e^{i\omega t - i\gamma z}$

$(\nabla^2+h^2)E_z=0$
$E_z \sim e^{i\gamma z}$
$(\nabla_{xy}^2+\lambda^2)E_z=0$
$\lambda^2=h^2-\gamma^2$

$\lambda^2 = h^2-\gamma^2$. $h = \frac{\omega}{c}$.

Get for E_z:

E-wave

$$E_z = A \sin\frac{\pi}{a} m x \, \sin\frac{\pi}{a} n x$$ [m, n integers ...

B.C.: $E_z = 0$ for: $x=0$, $x=a$; $y=0$, $y=b$.

This gives: $E_x \sim \frac{\partial E_z}{\partial x} \sim \cos..x \sin..y$

$E_x = 0$ for $y=0$, $y=b$ as it should.

$E_y \sim \frac{\partial E_z}{\partial y} \sim \sin..x \cos..y$

$E_y = 0$ for $x=0$, $x=a$ as it must

Get also: $\lambda = \pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$.

solutions described by m + n: E_{mn}

[Cylindrical case: "quantum no's" were n (order of Bessel fn) and m (m-th root of Bessel)

H-wave: $H_z = A \cos\frac{\pi m x}{a} \cos\frac{\pi n y}{b}$.

then $E_x \sim \frac{\partial H_z}{\partial y}$ vanishes for $y=0, b$ ✓

$E_y \sim \frac{\partial H_z}{\partial x}$ " " $x=0, a$ ✓

$H_x \sim \sin x \cos y$
$H_y \sim \cos x \sin y$.

Here, cases $m=0$, or $n=0$ not excluded. $\lambda = \pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$

smallest possible λ, if $a>b$, is $\lambda_{min} = \frac{\pi}{a}$ (H_{10} wave).

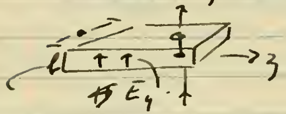
for E waves: $\lambda_{min} = \pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$ (E_{11} wave (or H_{11} wave))

H_{10} wave: $\quad H_z \sim \cos \frac{\pi x}{a}$

$$E_x = 0 \quad E_z = 0$$

$$E_y \sim \sin \frac{\pi x}{a}.$$

This is important wave; we can reduce b as much as we want, without changing a and hence λ_{min}:



Excite by electron beam:

transit time $= \frac{b}{v} \sim \pi/\omega = \frac{\pi}{c\lambda} = \frac{a}{c}$.

$\therefore$ need $a \gg b$.

$$b \sim a\left(\frac{v}{c}\right) \sim \frac{1}{30} a.$$

E wave:

require $\frac{b}{v} \sim \frac{b}{c}$, $\quad \therefore$ can only excite H_{10} wave by electrons.

(practically) impossible.

{ Power consumption: $\int H_{tan}^2 \, dS = \int_0^a H_{tan}^2 \, dx + \int_0^b H_{tan}^2 \, dx$

2 (both surfaces)

Do this as a problem.

treat same way as cylindrical case; find transmission S.

and find attenuation $\beta = \frac{P}{2S}$.

{ H_{10} wave here is more nearly like H_1 for cylindrical case;

J_0 (H_0-wave) corresponds more nearly to H_{11} wave here

Dec. 12 '41 From wave guides we may go to:

(1) two concentric cylinders (conducts even at low frequencies; at high freq. behaves like the hollow tube we discussed (except need both J + N sol'n's). E_{tan} then must vanish at inner tube as well as outer. Treated well in Stratton, E.M. Theory. Also a paper by Barrow &c (reprinted).

(2) Electromagnetic horns (wave guide of varying x-section

e.g. a conical shaped conductor; used as antenna; has advantage over conventional antenna in smaller losses. Will discuss later.

(3) Cavities.

Different from waveguide only in that z-dimension is of the same magnitude as others; hence also B.C. on bounding planes in z-direction; hence characteristic frequencies in 3-dimensions. Because they have resonant frequencies, can be used as radiators + detectors; problems are (1) characteristic frequencies (2) (3) the damping Q.

Consider a parallelepiped (rectangular); sides a, b, c. in x, y, z direction.

B.C.: $\begin{cases} E_x = 0 \text{ at } z=0,\ z=c,\ y=0,\ y=b. \\ E_y = 0 \text{ at } z=0,\ z=c,\ x=0,\ x=a \\ E_z = 0 \text{ at } x=0,\ x=a,\ y=0,\ y=b \end{cases}$

Hence
$$\begin{cases} E_x = A_x \cos\frac{\pi l x}{a} \sin\frac{\pi m y}{b} \sin\frac{\pi n z}{c} \\ E_y = A_y \sin\frac{\pi l x}{a} \cos\frac{\pi m y}{b} \sin\frac{\pi n z}{c} \\ E_z = A_z \sin\frac{\pi l x}{a} \sin\frac{\pi m y}{b} \cos\frac{\pi n z}{c} \end{cases}$$

But we have also $\nabla \cdot E = 0$;

whence $\frac{\pi l}{a} A_x + \frac{\pi m}{b} A_y + \frac{\pi n}{c} A_z = 0$

Let us call $\underline{k}$ the vector with components: $(\frac{\pi l}{a}, \frac{\pi m}{b}, \frac{\pi n}{c})$

i.e. $\underline{k}\cdot\underline{A} = 0$. i.e. $\underline{A} \perp \underline{k}$.

we could write each component in terms of eight waves $e^{\pm\frac{i\pi l x}{a} \pm \frac{i\pi m y}{b} \pm \frac{i\pi n z}{c}}$. (transverse waves).

Notice also that $k = \frac{\omega}{c}$. $[(\nabla^2 + \frac{\omega^2}{c^2})E_i = 0$ where $k^2 = \frac{\omega^2}{c^2}$.

$\frac{\omega^2}{c^2} = k^2 = \pi^2(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2})$. Can't have all three of $l, m, n = 0$.

Nor even two; not more than one $= 0$.

e.g. if $l=0$, $E_x \neq 0$, $E_y = 0 = E_z$, i.e. plane polarized standing wave.

Now we need the magnetic field:

$$-ik H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

$$= (k_x A_y - k_y A_x) \cos\frac{\pi l x}{a} \cos\frac{\pi m y}{b} \sin\frac{\pi n z}{c}$$

$$\therefore \underline{H} = \frac{i(\underline{k} \times \underline{A})_z}{k} \cos\cos\sin; \quad [\text{sin goes with cos-limits of } H \ldots$$

$\underline{k} \times \underline{A}$ is $\perp$ to k & A; $\therefore$ E, H, k, are mutually $\perp$.

$\nabla \cdot H = 0$ automatically;

$$\text{for } \nabla \cdot H = i[\underline{k} \cdot (\underline{k} \times \underline{A})] \frac{1}{k} \cos\cos\sin = 0.$$

Smallest value of k: if c is smallest dimension of box, then take $n=0$, $m=1$, $l=1$, and $k^2 = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$.

$$k = \frac{\pi}{ab}\sqrt{a^2+b^2}.$$

$$\text{Wavelength } \Lambda = \frac{2ab}{\sqrt{a^2+b^2}}.$$

Longest wave-length if $a=b$:

$$\Lambda_{max} = \sqrt{2}(a).$$

Dec. 15.

(Hansen + Perkins) Let P = power consumption/sec.

W = total e.m. energy in cavity.

$\beta = \frac{P}{2W}$; i.e. fields decay as $e^{-\beta t}$.

$W = \frac{1}{8\pi}\int(E^2+H^2)\,d\tau$. For a plane wave $\bar{E} = \bar{H}$; our sol'n is a superposition of plane waves, Hence $\int E^2 = \int \bar{H}^2$,

$$\text{and } W = \frac{1}{4\pi}\iiint \bar{H}^2\,d\tau.$$

Power consumption is $P = \frac{1}{\sigma}\int \underline{j}^2\, d\tau$

$$= \frac{1}{\sigma}\int j^2\, d\sigma\, dn$$

$\int j^2 dn = \frac{1}{D}\left(\int j\, dn\right)^2$ D = thickness of current sheath $D \approx \frac{1}{\sqrt{2\pi\sigma\mu\omega}}$

[Walls > .010 mm thick may be considered infinitely thick.]

Now $4\pi \int j\, dn = \underline{n} \times \underline{H}$

$j \sim e^{-\frac{n}{D}(1+i)}$

$|j|^2 \sim e^{-2\frac{n}{D}}$ $\longrightarrow$ $\int j\, dn = \frac{D}{1+i}$

$$\left|\int j\, dn\right|^2 = \frac{D^2}{2}.$$

$$\int j^2 dn = \frac{D}{2}.$$

$$\therefore P = \frac{1}{\sigma}\int \overline{j^2}\, d\tau$$

$$= \frac{1}{\sigma D}\int \overline{\left(\int j\, dn\right)^2}\, d\sigma$$

$$= \frac{1}{(4\pi)^2\sigma D}\int \overline{(n\times H)^2}\, d\sigma$$

$$= \frac{1}{(4\pi)^2\sigma D}\int \overline{H_{tan}^2}\, d\sigma$$

$$= \frac{2\pi\mu\omega D^2}{(4\pi)^2 D}\int \overline{H_{tan}^2}\, d\sigma$$

$$= \frac{\mu\omega D}{8\pi}\int \overline{H_{tan}^2}\, d\sigma$$

but $: h\; H_n = (\operatorname{curl} E)_n = 0$

$$\therefore \overline{P} = \frac{\mu\omega D}{8\pi}\int \overline{H}^2\, d\sigma$$

$$\beta = \frac{P}{2W} = \frac{\mu\omega D}{4}\frac{\int H^2 d\sigma}{\int H^2 d\tau} \sim \frac{\omega D}{L}.$$

$$Q = \frac{\omega}{2\beta} = \frac{2}{\mu D} \frac{\int H^2 d\tau}{\int H^2 d\sigma} \sim \frac{L}{D}.$$

For a rectangular cavity: let $l, m, n \neq 0$.

then $\int |H|^2 d\tau = \frac{1}{8} B^2 \cdot V.$ V = volume. $\underline{B} = i \frac{\underline{k} \times \underline{A}}{|k|}$.

$$\int_{xy\ \text{surface}} H^2 d\sigma = \frac{1}{2} B_x^2 a(b+c).$$

whole integral: $\frac{1}{2} B_x^2 a(b+c) + \frac{1}{2} B_y^2 b(a+c) + \frac{1}{2} B_z^2 c(a+b)$

$$\therefore \frac{1}{Q} = \frac{\mu D}{2} \cdot \frac{\frac{1}{2}}{\frac{1}{8} V} \left\{ \frac{B_x^2}{B^2} a(b+c) + \frac{B_y^2}{B^2} b(a+c) + \frac{B_z^2 c(a+b)}{B^2} \right.$$

$V = abc$ $$= \frac{2\mu D}{B^2} \left\{ B_x^2 \left(\frac{1}{b} + \frac{1}{c} \right) + \cdots \right\}.$$

for $B_x = B_y = B_z$, $a = b = c$,

$$\frac{1}{Q} = \frac{2\mu D}{B^2} \left\{ B_x^2 \left(\frac{2}{a} \right) + B_y^2 \left(\frac{2}{a} \right) + \cdots \right\}$$

$$= \frac{4\mu D}{a}, \quad Q = \frac{a}{4\mu D}.$$

If $n = 0$, $l, m \neq 0$, $\int |H|^2 d\tau = \frac{1}{4} B^2 V$

$$\int |H|^2 d\sigma = (B_x^2 + B_y^2) \cdot \frac{1}{2} ab$$

xy — $B_x^2\, ac$

yz — $B_y^2\, bc$

$$\therefore \frac{1}{Q} = \frac{\mu D}{2} \cdot \frac{B_x^2 (ac + \frac{ab}{2}) + B_y^2 (bc + \frac{1}{2} ab)}{\frac{1}{4} B^2 \cdot abc}$$

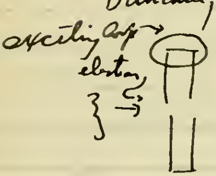
$$= \mu D \left\{ \frac{B_x^2}{B^2} \left(\frac{2}{b} + \frac{1}{c} \right) + \frac{B_y^2}{B^2} \left(\frac{1}{c} + \frac{2}{a} \right) \right\}$$

If $c \ll a, b$, $\frac{1}{Q} \sim \frac{D}{c}$ or $Q = c/D$. in general want large Q.

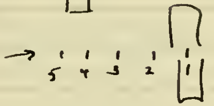
Problem: Study $H \times j$ distrib. for lowest mode with square base

Dec. 17. $Q \sim \frac{\text{dimension}}{D}$. Transit time for electron: $\leq \frac{1}{2}$ Period

"Bunching" – Varian, Webster, J. App. Phys. '39.



electrons are alternately accelerated & decelerated in cavity; results in groups of electrons coming out: (Field alternately with & against electron)



etc.

If these "bunched" electrons are sent thru another cavity, it will be excited; thus the oscillations are transmitted & can even be amplified quite considerably.

Transit time: if g is width of cavity: $= g/v$.

Period of oscillation $= \Lambda/c$.

$$\therefore \frac{g}{v} \leq \frac{\Lambda}{2c}$$

$$\therefore g \leq \Lambda \frac{v}{2c} = \Lambda \sqrt{\frac{E}{2mc^2}}$$

$2mc^2 = 1$ Mev.

if $E = 400$ v., $g \leq \Lambda \cdot \frac{20}{1000} = \frac{\Lambda}{50}$.

i.e. $g \leq$ other dimensions of cavity.

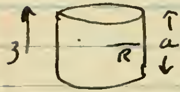
If we have, e.g., dimension a, a, g, with $g \ll a$, then the lowest mode is with $E \parallel g$; and $\Lambda = a\sqrt{2}$; (convenient).

$$\therefore g \leq a\sqrt{\frac{E}{mc^2}}$$

i.e. $g \sim \frac{1}{30} - \frac{1}{40}\, a$.

But this reduces Q. [want large Q for: sharp resonance, high signal to noise ratio.; & greater amplification. smaller power loss (if used as emitter).

Consider a cylindrical cavity:



get E waves: $E_z \neq 0$, $H_z = 0$

H waves: $H_z \neq 0$, $E_z = 0$.

For E waves: $E_z = A \, \frac{\pi l}{a} z \; J_n(\lambda r) \frac{\cos}{\sin} n\varphi$

(with $J_n(\lambda R) = 0$). (n, l, integers).

We had: =

$$\lambda^2 E_r = -i\left(\frac{\partial E_z}{\partial r} - \frac{il}{r}\frac{\partial H_z}{\partial \varphi}\right.$$

[$E_r = 0$ at $z = 0$, $z = a$ ∴ $\frac{\partial E_z}{\partial r} = 0$ at $z = 0, a$.

∴ $\frac{\partial E_z}{\partial r} \sim \sin z$ [E] wrong.]

$E_z = A \cos\frac{\pi l z}{a} \, J_n(\lambda r) \frac{\cos}{\sin} n\varphi$.

$E_r \sim \sin\frac{\pi l z}{a} \, J_n'(\lambda r) \frac{\cos}{\sin} n\varphi$

$E_\varphi \sim \sin\frac{\pi l z}{a} \, J_n(\lambda r) \frac{\sin}{\cos} n\varphi$. $n = 0, 1, 2 \ldots$

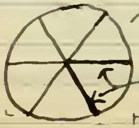
$H_r \sim \cos\frac{\pi l z}{a} \, J_n(\lambda r) \frac{\sin}{\cos} n\varphi$ n^{th} root of J

$H_\varphi \sim \cos\frac{\pi l z}{a} \, J_n'(\lambda r) \frac{\cos}{\sin} n\varphi$ $l = 0, 1, \ldots$

if $l = 0$, E is in z-direction – other comp. vanish. This corresponds to mode used with rectangular box on previous page. Take 1st root of J_0 also, then E_z falls from max at middle to zero at sides.

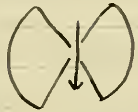
Can make $a \ll R$, & let electrons go thru in z-direction; not very different from previous problem —

But if $n \neq 0$, E_z has nodes where $\frac{\cos}{\sin} = 0$; Look at cross section of cylinder: let $n = 3$. consider nodes of $\cos 3\varphi$.



Can put a metal wall in where nodes are; situation isn't changed, since E_z vanishes there anyhow. Only effect is to lower Q.

Hence can make this shape: and can send



electrons thru narrow part (in plane of paper). So there is now a small dimension thru which to send electrons, and a large dimension to determine wave-length.

also $Q \sim \frac{\int |H|^2 d\tau}{\int |H|^2 d\sigma} \sim r$

In fact now n doesn't have to be integer;

Consider an H-wave: $H_z = \sin \frac{\pi l z}{a} J_n(\lambda r) \cos n\varphi$

$E_z = 0$

$E_\varphi = \quad J_n'(\lambda r) \cos n\varphi$

$E_r = \quad J_n(\lambda r) \sin n\varphi$

Also, we can use other Bessel f'n's, because we have left out the center.

Dec. 19. Above is not so. — can't get large oscillations by electrons. ———

Spherical Cavity:

Much work done on spherical elmag fields in optics, e.g. scattering of light by conducting spheres, etc.

[Mie ~ 1908]

[Debye - 1909]

[Born - Optik p. 275]

Maxwell eq'ns in polar coord's.

$$ik E_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta H_\varphi) - \frac{\partial}{\partial\varphi}(r H_\theta)\right].$$

$$ik E_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial H_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\sin\theta H_\varphi)\right]$$

$$ik E_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial\theta}\right].$$

$-ikH$ = same with E instead of H.

Distinguish between: E waves: $E_r \neq 0$, $H_r = 0$

H " : $H_r \neq 0$, $E_r = 0$

Boundary cond: $E_\varphi = E_\theta = 0$ at $r = R$.

E waves: $H_r = 0$, $\therefore \frac{\partial}{\partial\theta}(r\sin\theta\, E_\varphi) = \frac{\partial}{\partial\varphi}(r E_\theta)$.

Then ~~$-ik H_\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r E_\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial\theta}\right)$~~.

i.e. $\begin{cases} r E_\theta = \frac{\partial U}{\partial\theta} \\ r\sin\theta\, E_\varphi = \frac{\partial U}{\partial\varphi} \end{cases}$

E_θ eq'n:

$$ik E_\theta = \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta\, H_\varphi)$$

$$\therefore ik\sin\theta\,\frac{\partial U}{\partial\theta} = -\frac{\partial}{\partial r}(r\sin\theta\, H_\varphi) = \downarrow$$

Let $U = \frac{\partial}{\partial r}(r\Pi)$ $\Rightarrow$ $-ik\sin\theta\,\frac{\partial^2}{\partial\theta\partial r}(r\Pi)$

E_φ eq'n

$$\therefore H_\varphi = -\frac{ik}{r}\frac{\partial}{\partial\theta}(r\Pi) = -ik\frac{\partial\Pi}{\partial\theta}.$$

$$\therefore \frac{\partial}{\partial r}(r H_\theta) = \frac{ik}{\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}(r\Pi)$$

$$H_\theta = \frac{ik}{\sin\theta}\frac{\partial\Pi}{\partial\varphi}.$$

E_r eq'n. and $E_r = -\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta\frac{\partial\Pi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\frac{r}{\sin\theta}\frac{\partial\Pi}{\partial\varphi}\right]$.

$$= -\frac{1}{r}\Omega\Pi$$

[Ω is angular part of ∇^2 in sph. coord]

All quantities are now expressed in terms of Π.

Haven't used yet the H_θ & H_φ eq'n;

H_θ eq'n:

$$-ik H_\theta = \frac{1}{r\sin\theta}\left(\frac{\partial E_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\sin\theta\, E_\varphi)\right)$$

$$\therefore +\frac{k^2}{\sin\theta}\frac{\partial\Pi}{\partial\varphi} = \frac{1}{r\sin\theta}\left(-\frac{1}{r}\frac{\partial}{\partial\varphi}\Omega\Pi - \frac{\partial^2 U}{\partial\varphi\partial r}\right)$$

$$= \frac{-1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\Omega\Pi - \frac{\partial^3(r\Pi)}{\partial r\partial\varphi\partial r}\right)$$

$$\therefore \frac{\partial}{\partial\varphi}\left[k^2\Pi + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi) + \Omega\Pi\right] = 0 \quad \frac{\partial}{\partial\varphi}\left[k^2 + \nabla^2\right]\Pi = 0$$

H_φ eq'n:

$$-ik H_\varphi = \frac{1}{r}\left(\frac{\partial}{\partial r}(r E_\theta)\right) - \frac{\partial E_r}{\partial \theta}\Big]$$

$$-k^2 \frac{\partial \Pi}{\partial \theta} = \frac{1}{r}\left(\frac{\partial}{\partial r}\frac{\partial u}{\partial \theta} - \frac{\partial}{\partial \theta}\cdot\left(-\frac{1}{r}\Omega\Pi\right)\right)$$

~~$= \frac{1}{r}\frac{\partial^3}{\partial r^2 \partial\theta}(r\Pi)$~~

$$\therefore \frac{\partial}{\partial\theta}\left[k^2 + \nabla^2\right]\Pi = 0 \quad \text{again.}$$

If Π depends only on r, all components of field vanish; hence $(k^2+\nabla^2)\Pi = 0$ and Π satisfies wave eq'n.

B.C. $\quad \frac{\partial}{\partial r}(r\Pi)_{r=R} = 0$.

H wave: interchange E + H, + change sign of k.
Then B.C. is $\Pi = 0$ for $r = R$.

Solution of wave eq'n:

$$\Pi = \frac{\psi(r)}{r} P_{lm}(\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi$$

Equation for ψ: $\quad \frac{d^2\psi}{dr^2} + \left(k^2 - \frac{l(l+1)}{r^2}\right)\psi = 0$.

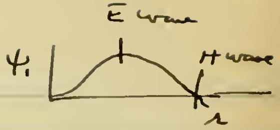
Solution: highbrow: $\psi \sim \sqrt{r}\, J_{l+1/2}(kr)$

lowbrow: $\sim \sin kr\, f_1(kr) + \cos kr\, f_2(kr)$

(f_1 + f_2 polynomials)

$l = 0$: no solution: because this gives $\Pi = \Pi(r)$ only, which gives no field. [i.e. no solutions to el'mag. eq'ns with vanishing angular momentum — by angular momentum is meant $\int(\underline{r} \times \underline{p})\,dV$; momentum $p \sim \underline{S}$ - (Poynting vector) turns out always $\sim l$.] .

$l = 1$: get $\psi = \frac{\sin kr}{kr} - \cos kr$.

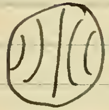


E wave sol'n: $kR \sim 2.5$

H wave sol'n: $kR \sim 4.0$

$l=1,\ m=0 \qquad \Pi = \frac{1}{r}\left(\frac{\sin kr}{kr} - \cos kr\right)\cos\theta$

then $E_\theta \sim \sin\theta \sim H_\varphi$



$E_\varphi = 0 = H_\theta$

$E_r \sim \cos\theta$

$l = 1,\ m = \pm 1$: same: get $\sin\theta \frac{\cos}{\sin}\varphi$ which means just replace z-direction by x or y direction.

Jan 5 1942 We had:

$E_\theta = \frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r\,\partial\theta}$

$E_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(r\Pi)}{\partial r\,\partial\varphi}$

$H_\varphi = \frac{ik}{r}\frac{\partial(r\Pi)}{\partial\theta}$

$H_\theta = -\frac{ik}{r\sin\theta}\frac{\partial(r\Pi)}{\partial\varphi}$.

$E_r = -\frac{1}{r}\Omega\Pi$

$H_r = 0$

$\nabla^2(\Pi r) = 0$

$\therefore \Pi = \frac{\psi(r)}{r} P_{lm}(\theta)\frac{\cos}{\sin}m\varphi$.

E waves:

$\left(\frac{\partial\psi}{\partial r}\right)_R = 0$.

H waves:

interchange E & H in equations, & replace k by $-k$.

and $\psi = 0$ at $r = R$

In general $\psi \sim \sqrt{r}\, J_{l+\frac{1}{2}}(kr)$.

We found mode $l=0$ doesn't exist.

$l=1$: see top of page.

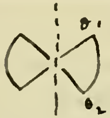
these solutions have same form as 2^l-pole radiation

i.e. $l=1$: E-wave $\rightarrow$ electric dipole

$l=1$: H-wave $\rightarrow$ magnetic dipole.

[in case of radiation, assume radiating charges etc. at origin, & use Hankel function instead of Bessel f'n: $\psi \sim \sqrt{r}\, H^{(1)}_{l+1/2}(r)$

Consider sphere with cone cut out.



Old solutions still hold; but also get new ones, because poles are not included; hence solutions don't have to behave at $\theta = 0$ & π; in this case the Legendre equation has solutions Q_l.

$$Q_0 \sim \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta}$$

$$Q_l \sim P_l Q_0 + (\cos\theta)^{l-1} + \dots$$

Now the mode $l=0$ is no longer missing, since (Π/r) is no longer independent of θ for $l=0$.

$l=0$: $\quad r\Pi = -\sin kr\; Q_0(\theta)$

since $r\Pi$ is indep. of φ, we get $E_\varphi = 0 = H_\theta$

but $E_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}(r\Pi)$

$$= +\frac{1}{r}\cdot k\cos kr\cdot\frac{1}{2}\left(\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\right)(-\sin\theta)$$

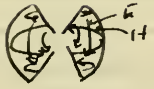
$$= +\frac{k}{r}\frac{\cos kr}{\sin\theta}.$$

$$H_\varphi = -\frac{ik}{r}\sin kr\cdot -\frac{1}{\sin\theta}$$

$$= \frac{ik\sin kr}{r\sin\theta}.$$

$$E_r = \frac{1}{r}\Omega(\Pi r) = \frac{1}{r}\Omega(\psi Q_l) = l(l+1)\frac{\psi}{r}.$$

$\therefore\ l=0:\ E_r = 0$



$E_\theta = 0$ at surface:

$\cos kR = 0, \quad kR = \pi/2 = 1.57$

∴ frequency is lower for this mode than for any of those of whole sphere.

On cones: $E \perp$ surface, hence B.C. satisfied.

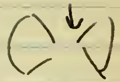
for $l \neq 0$, E_r in general $\neq 0$; must make $E_r(\theta_1) = 0$. this will require a linear combination $E_r(\theta_2) = 0$.
$P_l + \alpha Q_l$. and in general not any θ_1 & θ_2 will be possible. — will be related.

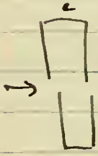
$l=0$: $\int_{\theta_1}^{\theta_2} E_\theta \cdot r\,d\theta = \cos kr \ln \tan \theta/2 \Big|_{\theta_1}^{\theta_2}$
∴ maximum "potential difference" at center,
∴ electron thru center has maximum coupling.

Problem: Calculate "Q" of cavity $\int H_{tan}^2$.

Jan. 7'42 Q for spherical cavity with holes:

$$Q \sim \frac{\int H^2 dV}{D \int H^2 d\sigma} \approx \frac{V}{AD} \sim \frac{R}{D} .$$



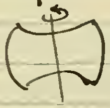


[for rectangular cavity, we had, $Q \sim \frac{c}{D}$

$R \gg c$, ∴ Q much larger for spherical cavity.

[actually Q is considerably less than $\frac{R}{D}$; but still better than rectangular cavity]

Hansen & Richtmyer also calculate for a cavity made out of sections of ellipse & hyperbolae of rotation:



General theory of oscillations in a cavity — Condon, J. App. Ph. 1940-41.

Given a general cavity; $\underline{E}$ & $\underline{H}$ can then be calculated from Maxwell's equations & boundary conditions:

$$\begin{cases} c\nabla\times\underline{E} = -\dot{\underline{H}} \\ c\nabla\times\underline{H} = \dot{\underline{E}} \\ \nabla\cdot\underline{H} = 0 \\ \nabla\cdot\underline{E} = 0 \\ \underline{E}_{tan} = 0 \text{ at surface} \end{cases}$$

Introduce vector potential $\underline{A}$:

$$\underline{H} = \nabla\times\underline{A}$$ no scalar pot.

$$\underline{E} = -\tfrac{1}{c}\dot{\underline{A}}$$ - no charges.

Spatial dependence of A is same as E for any normal mode (i.e. (harmonic time dependence.)

$\therefore\ \nabla\cdot\underline{A} = 0$ for normal modes.

A satisfies: $\nabla^2\underline{A} = +\frac{1}{c^2}\ddot{\underline{A}}$

for a normal mode $A_m(t) = A_m e^{i\omega_m t}$

$$\nabla^2 A_m = -k_m^2 A_m \quad \text{with } k_m = \frac{\omega_m}{c}.$$

$$\text{or } (\nabla^2 + k^2)A_m = 0.$$

subject to $\underline{A}_{tan} = 0$ at surface

Treat as wave-mechanical problem; A_m are wave-functions, k_m are eigenvalues.

; Are A_m orthogonal?

~~$\int \underline{A}_\ell \cdot \underline{A}_m dV$~~

$$\int(\underline{A}_\ell\cdot\nabla\times\nabla\times A_m - \underline{A}_m\cdot\nabla\times\nabla\times\underline{A}_\ell)dV = (k_m^2 - k_\ell^2)\int\underline{A}_\ell\cdot\underline{A}_m\,dV$$

Left hand side can be made a surface integral;

$$\nabla\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot\nabla\times\underline{A} - \underline{A}\cdot\nabla\times\underline{B}$$

$$\therefore\ \nabla\cdot(\underline{A}_\ell\times\nabla\times\underline{A}_m) = \nabla\times\underline{A}_m\cdot\nabla\times\underline{A}_\ell - \underline{A}_\ell\cdot\nabla\times\nabla\times\underline{A}_m$$

$$\nabla\cdot(\underline{A}_m\times\nabla\times\underline{A}_\ell) = \nabla\times\underline{A}_m\cdot\nabla\times\underline{A}_\ell - \underline{A}_m\cdot\nabla\times\nabla\times\underline{A}_\ell$$

$$\therefore \text{ Left side above } = \int(A_\ell\times\nabla\times A_m - A_m\times\nabla\times A_\ell)_n\,dS$$

To know normal comp of $(A \times B)$, need tangential components of $A \times B$; since $\underline{A_m}_{tan} = 0 = \underline{A_\ell}_{tan}$, the surface integral vanishes;

Hence $(k_m^2 - k_\ell^2) \int \underline{A_\ell} \cdot \underline{A_m} = 0$

Hence $\int \underline{A_\ell} \cdot \underline{A_m} = 0$ unless $k_m = k_\ell$.

i.e. two non-degenerate eigenfunctions are orthogonal.

[Conv, in case of degeneracy, find solutions which are orthogonal — linear combinations of original solutions.]

[In the cases we have treated, we had orthogonal degenerate eigenf'ns : spherical cavity, P_ℓ^m are degenerate with resp. to m, but P_ℓ^m are orthogonal].

Normalization : we will use $\int_V A_m^2 \, dV = V$

Example : Rectangular cavity:

We had $E_x = C_x \cos \frac{\pi n_1 x}{L_1} \sin \frac{\pi n_2 y}{L_2} \sin \frac{\pi n_3 z}{L_3} = A_x$

$L_1 \, L_2 \, L_3$ = edges of cavity, $n_1 \, n_2 \, n_3$ integers.

then $\int A^2 dV = C^2 V \cdot \frac{1}{8}$ $\qquad C^2 = C_x^2 + C_y^2 + C_3^2$.

$\therefore C^2 = 8$ to get desired normalization

[also $\underline{C} \cdot \underline{k} = 0$].

If $n_1 = 0$ $\quad A_y = 0 = A_3$

and $\int A^2 dV = \frac{1}{4} C_x^2 V$ & $C_x = 2$.

$\therefore$ normalized eigenf'n is $A_x = 2 \sin \frac{\pi n_2 y}{L_2} \sin \frac{\pi n_3 z}{L_3}$

Energy : $W = \frac{1}{8\pi} \int (E^2 + H^2) \, dV$.

Vector potential can be expanded in terms of normal modes:

$$\underline{A} = \sum_m p_m(t)\underline{A}_m(x)$$

$$-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\nabla^2 A$$

$$\therefore \frac{1}{c^2}\sum \ddot{p}_m(t) A_m = \sum p_m \nabla^2 A_m$$
$$= \sum p_m(-k^2 A_m)$$

Since A_m are orthogonal, A_m must have same coeff. on each side:

$$\frac{1}{c^2}\ddot{p}_m = -k^2 p_m$$

$$\ddot{p}_m + \omega^2 p_m = 0$$

whence $p_m = \overset{\cos}{\sin}\ \omega_m t$.

Dissipation term may be added — proportional to $\dot{p}_m$: then get:

$$\ddot{p}_m + \frac{\omega_m}{Q_m}\dot{p}_m + \omega_m^2 p_m = 0$$

To show this is correct form, will calculate energy + energy loss.

This is for free oscillation; will also discuss forced oscillation (loop, or electron beam, thru cavity).

Now we have:

$$\underline{E} = -\frac{1}{c}\sum \frac{dp_m}{dt}\cdot \underline{A}_m$$

$$\underline{H} = \sum p_m \operatorname{curl} \underline{A}_m$$

then

$$W = \frac{1}{8\pi}\sum_l \sum_m \left(\frac{1}{c^2}\dot{p}_l \dot{p}_m \int dV\, A_l\cdot A_m + p_l p_m \int dV \operatorname{curl} A_l \cdot \operatorname{curl} A_m\right)$$

$$= \frac{1}{8\pi}\sum_l \sum_m \left(\dot{p}_l \dot{p}_m \frac{\delta_{lm}}{c^2} V + p_l p_m \int dV\, A_l\cdot \operatorname{curl}\operatorname{curl} A_m\right)$$

$$\int dV A_l \cdot k^2 A_m \rightarrow k^2 V\cdot\delta_{lm}$$

$$\therefore W = \frac{1}{8\pi}\sum_m \left(\frac{\dot{p}_m^2 + \omega_m^2 p_m^2}{c^2}\right)V = \frac{V}{8\pi c^2}\sum\left(\dot{p}_m^2 + \omega^2 p_m^2\right)$$

Jan. 9, '42 We found solutions $\underline{A}_m$ such that $\int \underline{A}_l \cdot \underline{A}_m \, dV = V \delta_{lm}$
We expanded $\underline{A} = \Sigma \, p_m(t) \, \underline{A}_m(r)$
and found that p_m satisfy $\ddot{p}_m + \omega^2 p_m = 0$.
Then $\underline{E} = -\frac{1}{c}\Sigma \, \dot{p}_m \, \underline{A}_m$, $H = \Sigma \, p_m \, \nabla \times \underline{A}_m$.

We found further: $\int (\nabla \times A_l) \cdot (\nabla \times A_m) \, dV = k_m^2 \, V \delta_{lm}$
Hence $\nabla \times \underline{A}$ is as good an eigenf'n as A.
If we write $\nabla \times \underline{A}_l = k_l \, \underline{F}_l$
then $\int \underline{F}_l \cdot \underline{F}_m \, dV = V \delta_{lm}$ just like A's.

Now $\nabla \times \underline{F}_l = \frac{1}{k_l} \nabla \times \nabla \times A_l = \frac{1}{k_l} \cdot k_l^2 \underline{A} = k_l A$.
So F & A have a symmetrical, reciprocal relationship

We had also $W = \frac{1}{8\pi} \int (E^2 + H^2) dV = \frac{V}{8\pi} \sum_m \left(\frac{1}{c} \dot{p}_m^2 + k_m^2 p_m^2 \right)$
$= \frac{V}{8\pi c^2} \Sigma (\dot{p}_m^2 + \omega^2 p_m^2)$

If damping, $\ddot{p}_m + \frac{\omega_m}{Q_m} \dot{p}_m + \omega^2 p_m = 0$
if no damping: $\therefore \underline{p_m = \frac{\cos}{\sin} \omega_m t}$
$\overline{\dot{p}_m^2} = \omega_m^2 \overline{p_m^2}$
$\therefore W_{el} = W_{mag}$.

If E, H are given at t = 0,
$\int \underline{E} \cdot \underline{A}_m \, dV = -\frac{1}{c} \frac{dp_m}{dt} \cdot V$
and $\int \underline{H} \cdot \underline{F}_m \, dV = k_m p_m V$
} and can find initial values of p's & $\dot{p}$'s.

or $\begin{cases} \dot{p}_m(0) = -\frac{c}{V} \int \underline{E(0)} \cdot \underline{A}_m \, dV \\ p_m(0) = \frac{1}{k_m V} \int \underline{H(0)} \cdot \underline{F}_m \, dV \end{cases}$

Dissipation. $\ddot p + \frac{\omega_m}{Q_m}\dot p + \omega_m^2 p = 0$

$$\ddot p \dot p + \frac{\omega_m}{Q_m}\dot p^2 + \omega_m^2 p \dot p = 0$$

$$\therefore \tfrac{1}{2}\dot p^2 + \int \frac{\omega_m}{Q_m}\dot p^2\,dt + \tfrac{1}{2}\omega_m^2 p^2 = C.$$

$$\therefore \quad W = \tfrac{1}{2}(\dot p^2 + \omega_m^2 p^2) = C - \frac{\omega_m}{Q_m}\int \dot p^2\,dt$$

so energy decreases with time.

$$\therefore \frac{dW}{dt} = -\frac{\omega_m}{Q_m}(\dot p)^2 = P$$ consumption power per ~~unit time~~

We had $P \sim D\nu \int H^2\,d\sigma$

But $\int H^2 d\sigma = \sum_{l,m} p_l p_m \int (\nabla\times A_l)(\nabla\times A_m)\,d\sigma$

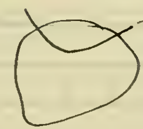
$$= \sum_m p_m^2 k_m^2 \int F_m^2\,d\sigma$$

For time averages: $\overline{p_m^2}\, k_m^2 = \frac{1}{c^2}\overline{\dot p^2}$

$$\therefore \; -\frac{\omega_m}{Q_m}\overline{\dot p^2} = D\nu \sum_m \overline{p_m^2}\, k_m^2 \int F_m^2$$

$$\therefore \frac{\omega_m}{Q_m} = D\nu \sum_m k_m^2 \int F_m^2\,d\sigma$$

Excitation.



→ conductor thru cavity.

use current with freq. close to cavity freq.

Condn does this:

$$\nabla\times \underline{H} = \frac{1}{c}\dot{\underline{E}} + 4\pi \underline{j}$$ $\underline{j}$ in e.m.u.

$$\nabla\times\nabla\times \underline{A} = -\frac{1}{c^2}\ddot A + 4\pi j$$

put in expansion: $\sum_m\left(\nabla\times\nabla\times A_m + \frac{1}{c^2}A_m \ddot p_m\right) = 4\pi j$

$$\therefore \quad \sum_m p_m k_m^2 A_m + \frac{1}{c^2} \ddot{p}_m A_m = 4\pi \underline{j} \; .$$

Expand $j_m = \sum_m A_m \frac{\int dV \, j \cdot A_m}{V}$.

Hence $p_m k_m^2 + \frac{1}{c^2} \ddot{p}_m = \frac{4\pi}{V} \int \underline{j} \cdot \underline{A_m} \, dV$.

or $\ddot{p}_m + \omega_m^2 p_m = \frac{4\pi c^2}{V} \int \underline{j} \cdot \underline{A_m} \, dV$.

If also dissipation: $\ddot{p}_m + \frac{\omega_m}{Q_m} \dot{p} + \omega_m^2 p_m = \frac{4\pi c^2}{V} \int \underline{j} \cdot \underline{A_m} \, dV$

def: $= L p_m$

If current is in a wire of small cross section, or small beam of electrons Here can integrate first over cross section of beam, then along whole beam;
(gives total current)

$$\therefore L p_m = \frac{4\pi c^2}{V} I \int_{\text{wire}} \underline{A_m} \cdot \underline{ds} \qquad = (\quad) M_m$$

if loop is completed along outside, $\oint \underline{A_m} \cdot \underline{ds} = \int (\nabla \times \underline{A_m}) \cdot \underline{d\sigma}$
(contribution from outside = 0)

$= \int \underline{H_m} \cdot d\sigma$

= flux thru loop due to unit excitation of mode m of cavity

If $I = I_0 e^{i\omega t}$
where $\omega \neq$ any ω_m

From ~~theory of~~ forced oscillation:

$$p_m = \frac{4\pi c^2}{V} M_m I_0 e^{i\omega t} \frac{1}{\omega_m^2 - \omega^2 + \frac{i\omega\omega_m}{Q_m}} \; .$$

+ resonance at cavity frequencies.

For general excitation $I(t)$.

get then from theory of 2nd order eq'ns:

$$p = \frac{4\pi c^2}{V} M_m \int_{-\infty}^{t} I(t') \sin \omega_m (t-t') \, e^{-\frac{\omega_m}{2Q}(t-t')} \, dt' \; .$$

Jan 12, '42 For excitation: $\ddot{p}_m + \frac{\omega_m}{Q_m}\dot{p}_m + \omega_m^2 p_m = \frac{4\pi c^2 I}{V}\int \underline{A}_m \cdot d\underline{s}$

Want $\int \underline{A}_m \cdot d\underline{s}$ to be as large as possible

Rectangular cavity:

integral relatively small.

integral large, field large at center [$A \sim \frac{\cos kr}{r}\frac{1}{\sin\theta}$

Impedance produced by cavity in circuit of exciter.

each mode is excited:

$$p_m = \frac{4\pi c^2 I \int \underline{A}_m \cdot d\underline{s}}{\omega_m^2 - \omega^2}\, e^{i\omega t}$$

now $E = -\frac{1}{c}\sum \dot{p}_m \underline{A}_m = -\frac{1}{c}\sum i\omega p_m \underline{A}_m$

then $\frac{\int \underline{E}\cdot d\underline{s}}{I}$ = impedance = R

$$R = \frac{1}{cI}\sum p_m^* i\omega \int \underline{A}_m \cdot d\underline{s}$$

$$= -\frac{4\pi c^2 I}{cI}\, i\omega \sum \frac{\left(\int \underline{A}_m \cdot d\underline{s}\right)^2}{\omega_m^2 - \omega^2}$$

or $R_{e.m.u.} = +i\omega 4\pi \sum \frac{\left(\int \underline{A}_m \cdot d\underline{s}\right)^2}{\omega_m^2 - \omega^2}$. (should be)

for $\omega \ll \omega_m$, $R \sim +i\omega$ inductance

$\omega \sim \omega_m$ resonances.

Bunching:

two [illegible]:

(I) → (II)   energy transfer to electron in 1st cavity:

$$e V \sin \omega t \, .$$

$$E = p_m e^{i\omega t} \underline{A}_m(r)$$

$$V = \int \underline{E} \cdot d\underline{s} = p_m e^{i\omega t_0} \int A_{m_x}(x)\, e^{i\omega x/v}\, dx$$

← retardation –

Simplest case: $A_{m_x} = \text{const}.$

have $\quad A_{m_x} \int_{-b/2}^{b/2} e^{i\omega x/v}\, dx$

$$= A_{m_x} \cdot \frac{e^{i\omega \frac{b}{2v}} - e^{-i\omega \frac{b}{2v}}}{\frac{i\omega}{v}}$$

$$= A_{m_x} \cdot b \cdot \frac{\sin \frac{\omega b}{2v}}{\frac{\omega b}{2v}} \, .$$

$\therefore$ want $\quad \frac{\omega b}{2v} < \pi/2$ ~~[illegible]~~

$$\frac{b}{v} < \pi/\omega = \tfrac{1}{2} \text{ period}$$

$$\tfrac{1}{2} m v^2 = \tfrac{1}{2} m v_0^2 + e V \sin \omega t$$

$$v^2 = v_0^2 + \frac{2eV}{m} \sin \omega t$$

or $\quad v = \left(v_0^2 + \frac{2eV}{m} \sin \omega t\right)^{1/2} \cong v_0 + \frac{eV}{m v_0} \sin \omega t$

(if ↑ is small)

$$x(t') = v(t' - t)$$

Second cavity at $x = a$. $\quad \therefore t' - t = \frac{a}{v} = \frac{a}{v_0} - \frac{a e V}{m v_0^3} \sin$ [illegible]

Excitation of second cavity:

$$p_2(t'') = \int^{t''} I(t') \sin \omega (t'' - t')\, e^{-\frac{\omega}{Q}(t'' - t')}\, dt'$$

if $I(t')$ is constant, $\quad p_2(t'') \sim 0 \, .$

but $I(t')$ is not constant, because of bunching — first many, then no electrons reach it — in rapid succession

$$I(t')\,dt' = I(t)\,dt \qquad \text{but } I(t) \sim \text{const.} \quad (1^{st} \text{ cavity})$$

$$\therefore p_2(t'') = \frac{Q}{\omega}\int_0^{t''} I(t)\,dt \,\sin\omega\left(t'' - t - \frac{a}{v_0} + \frac{aeV}{mv_0^3}\sin\omega t\right)\frac{Q}{\omega} \cdot e^{-\frac{\omega}{Q}(t''-t-a/v_0)}$$

write integral as:

$$\sin\omega\left(t'' - a/v_0\right)\cos\omega\left(t - \frac{aeV}{mv_0^3}\sin\omega t\right)$$
$$-\cos \quad '' \qquad \sin \quad ''$$

exponential is essentially constant —

$\therefore$ need

$$\int_{\omega t=0}^{2\pi} \begin{Bmatrix}\cos \\ \sin\end{Bmatrix} \omega\left(t - \frac{aeV}{mv_0^3}\sin\omega t\right) d(\omega t)$$

$$= \begin{Bmatrix} 2\pi \\ 0 \end{Bmatrix} J_1\left(\frac{aeV\omega}{mv_0^3}\right) .$$

$$\therefore \quad p_2(t'') = I \sin\omega\left(t'' - \frac{a}{v_0}\right) J\left(\frac{aeV\omega}{mv_0^3}\right)\cdot\frac{Q}{\omega^2}$$

$\therefore$ excitation of second cavity $\sim I$ (electron current)
$\sim \sin\omega t''$ same freq.
$\sim Q$ so large Q desirable
$\sim J_1$

J_1 has max when arg. $= 1.83$

$$\therefore aeV\omega = 1.83\, mv_0^3$$

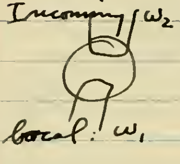
gives max. or: $a = \frac{1.83\, mv_0^2}{eV}\cdot\left(\frac{v_0}{\omega}\right)$

$$\simeq \frac{E_0}{eV}\cdot b \cdot 1.1$$

$$\therefore \frac{a}{b} \sim \frac{E_0}{eV} .$$

But then p_2 depends on I,
not on field in 1st cavity —
so arbitrary amplification.
(but need signal large compared with shot effect noise)

Jan 14, '42 As detector: Let signal & a local oscillator both excite cavity:



$\omega_1 \approx \omega_2$. Now we have, in previous equation

$$e V_1 \sin \omega_1 t + e V_2 \sin \omega_2 t$$
$$= e V_1 \sin \omega_1 t + e V_2 \left[\sin \omega_1 t \cos(\omega_2 - \omega_1 t) + \cos \omega_1 t \sin \ldots \right.$$
$$= e \sin \omega_1 t \left[V_1 + V_2 \cos(\omega_2 - \omega_1) t \right] + \cos \omega_1 t \sin(\omega_2 - \omega_1 \ldots$$
$$= e \sin\left[\omega_1 t + \frac{V_2}{V_1} \sin(\omega_2 - \omega_1) t\right] \left[V_1 + V_2 \cos(\omega_2 - \omega_1) t \ldots \right.$$

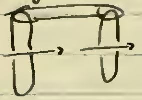
And the integral becomes: $J_1 \left(\frac{a e \omega}{m v_0^3} \left(V_1 + V_2 \cos(\omega_2 - \omega_1) t \right) \right) \sin \omega_1 \ldots$

Make $J_1' \left(\frac{a e \omega}{m v_0^3} V_1 \right)$ a maximum:

then get $J_1 \left(\frac{a e \omega V_1}{m v_0^3} \right) + \frac{a e \omega}{m v_0^3} V_2 \cos(\omega_2 - \omega_1) t \, J_1 \ldots$

constant ↗ detectable ↖ .

Klystron:



shot effect starts it; afterwards, second cavity is coupled to first, which bunches electrons which excite second cavity.

Emission of Radiation.

Cf. Stratton esp. applications

Use φ, $\underline{A}$, scalar & vector potential resp.
then $\underline{H} = \nabla \times \underline{A}$, $E = -\frac{1}{c}\dot{\underline{A}} - \nabla\varphi$
[then $\nabla \times \underline{E} = -\frac{1}{c}\nabla\times\dot{A} - \nabla\times\nabla\varphi = -\frac{1}{c}\dot{\underline{H}}$ ✓
and $\nabla\cdot H = \nabla\cdot\nabla\times A \equiv 0$ automatically.]

We also have: $\nabla\cdot\underline{E} = 4\pi\rho$, $\nabla\times H = +\frac{1}{c}\dot{\underline{E}} + \frac{4\pi}{c}\underline{j}$.

these give: $-\frac{1}{c}\nabla\cdot\dot{\underline{A}} - \nabla\cdot\nabla\varphi = 4\pi\rho$; $\nabla\times\nabla\times A = -\frac{1}{c^2}\ddot{\underline{A}} - \frac{1}{c}\nabla\dot{\varphi} + \frac{4\pi}{c}j$

If we impose the condition $\left\{\nabla\cdot\underline{A} + \frac{1}{c}\dot{\varphi} = 0,\right.$
then $\begin{cases}\nabla^2\varphi - \frac{1}{c^2}\ddot{\varphi} = -4\pi\rho \\ \nabla^2\underline{A} - \frac{1}{c^2}\ddot{\underline{A}} = -\frac{4\pi}{c}j\end{cases}$ $\quad \nabla\cdot\underline{j} + \frac{\partial\rho}{\partial t} = 0$

Which have sol'n's: $\varphi(\underline{r}, t) = \int d\underline{r}' \, \dfrac{\rho\left(\underline{r}', t - \frac{|r-r'|}{c}\right)}{|\underline{r} - \underline{r}'|}$

"retarded" potential.

$$\underline{A}(\underline{r}, t) = \int d\underline{r}' \, \frac{\underline{j}\left(\underline{r}', t - \frac{|r-r'|}{c}\right)}{c\,|r - r'|}$$

Can show from these
that $\nabla\cdot\underline{A} + \frac{1}{c}\dot{\varphi} = 0$, so that condition was OK.

Since j & ρ are connected, should be able to get φ from $\underline{A}$; this was done in older literature by introducing $\underline{\Pi}$ - the Hertzian vector; then $\underline{A} = \dot{\underline{\Pi}}$, $\varphi = \nabla\cdot\underline{\Pi}$

We will assume immediately a harmonic time dependence: $e^{-i\omega t}$ [then waves traveling, say, in + z direction have e^{+ikz}].

Then $\begin{cases}\underline{j}(\underline{r}', t) = \underline{j}_0(\underline{r}')\, e^{-i\omega t} \\ \rho(\underline{r}', t) = \rho_0(\underline{r}')\, e^{-i\omega t}\end{cases}$. then $\nabla\cdot\underline{j}_0 - i\omega\rho_0 = 0$

$\rho_0 = \frac{1}{i\omega}\nabla\cdot\underline{j}_0$.

this gives : $\varphi(\underline{r}, t) = \int \frac{dr'}{|r-r'|} \rho_0(\underline{r}') e^{-i\omega[t - \frac{|r-r'|}{c}]}$

$= \frac{c\, e^{-i\omega t}}{i\omega} \int \frac{dr'}{|r-r'|} \underline{\nabla} \cdot \underline{j}_0 \; e^{ik|r-r'|} \qquad k = \frac{\omega}{c}.$

and $\underline{A}(\underline{r}, t) = \frac{1}{c} e^{-i\omega t} \int \frac{dr'}{|r-r'|} \underline{j}_0(\underline{r}') e^{ik|r-r'|}.$

i.e. $= \frac{e^{-i\omega t}}{i\omega} \int \frac{dr'}{t} \underline{j}_0(\underline{r}') \cdot \text{grad}_r \frac{e^{ik|r-r'|}}{|r-r'|}$

$\cong \nabla \cdot \left\{ \int dr' \, \underline{j}_0(\underline{r}') \cdot \frac{e^{ik|r-r'|}}{|r-r'|} \right.$

i.e. $\varphi = \frac{1}{ik} \underline{\nabla} \cdot \underline{A}$.
[this comes also from $\nabla \cdot A + \frac{1}{c}\dot{\varphi} = 0$] .

If currents are confined to a region small comp and with wavelength, get practically unretarded result for points near it. i.e. essentially static fields.
At large distances, results again simple - so-called "wave zone"
At intermediate regions - rather complicated.

Wave zone : $r \gg r'$ Field at distance >> dimension of radiator.
also $r \gg \lambda$.

then neglect $\underline{r}'$ comp. with $\underline{r}$ in denominator;
in exp. write : $|\underline{r} - \underline{r}'| = r - \underline{s} \cdot \underline{r}'$ $\quad \underline{s}$ = unit vector in direction of r

then $\underline{A}(\underline{r}, t) = \frac{e^{-i\omega t}}{cr} \int dr' \, j_0(r') e^{ik(r - \underline{s} \cdot \underline{r}')}$

$= \frac{1}{cr} e^{i(kr - \omega t)} \int dr' \, \underline{j}_0(r') \, e^{-ik \underline{s} \cdot \underline{r}'}$.

Define: $\underline{J}(\underline{s}) = \int d\underline{r}'\, \underline{j}_0(\underline{r}') e^{-ik\underline{s}\cdot\underline{r}'}$

Then $\underline{A}(\underline{r}, t) = \frac{1}{cr} e^{i(kr-\omega t)} \underline{J}(\underline{s})$.

Jan 16, '42 then $\varphi = \frac{1}{ik} \nabla\cdot\underline{A} = \frac{1}{ik}\cdot\frac{e^{-i\omega t}}{cr}(\nabla e^{ikr})\cdot\underline{J}$ [assuming e^{ikr} strongest r-dependence]

$= \frac{1}{ik}\frac{e^{-i\omega t}}{cr}\cdot ik\, \underline{s}\cdot\underline{J}\, e^{ikr}$

$= \frac{e^{ikr-i\omega t}}{cr}\underline{s}\cdot\underline{J} = \underline{s}\cdot\underline{A}$.

And $\underline{H} = \nabla\times\underline{A} = \frac{1}{cr}e^{-i\omega t}(\nabla e^{ikr})\times\underline{J}$

$= \frac{1}{cr}e^{i(kr-\omega t)}\cdot ik\, \underline{s}\times\underline{J}$

$= \frac{ik}{cr}e^{i(kr-\omega t)}\underline{s}\times\underline{J}$.

Likewise $\underline{E} = -\frac{1}{c}\dot{\underline{A}} - \nabla\varphi = -\frac{1}{c}\cdot -i\omega\underline{A} - ik\,\underline{s}\,\underline{s}\cdot\underline{A}$

$= ik\,\underline{s}\times(\underline{A}\times\underline{s})$.

Hence E, H both ⊥ $\underline{s}$, direction of propagation.
also E ⊥ H, and in same phase.
outgoing spherical waves; phase propagated outward

If $\underline{J}$ is measured in amperes:

$\underline{E} = \frac{ik}{10}\left\{\frac{e^{i(kr-\omega t)}}{r}\underline{s}\times(\underline{J}\times\underline{s})\right\}$ e.s.u.

$= 30ik\{\quad\}$ volts/cm.

Energy flux: Poynting vector $\underline{S} = \frac{c}{4\pi}\underline{E}\times\underline{H}$

$\overline{S} = \frac{c}{4\pi}\cdot\frac{1}{2}|\underline{E}\times\underline{H}^*|$

$$\underline{E} \times \underline{H}^* = k^2 \cdot \frac{1}{r^2} \left[[\underline{s} \times (\mathcal{J} \times s)] \times (\mathcal{J}^* \times s) \right]$$

$$= \frac{k^2}{r^2} \, \underline{s} \, |\mathcal{J} \times \underline{s}|^2 .$$

$$\therefore \; \overline{\underline{S}} = \frac{c}{8\pi} \frac{k^2}{r^2} \overline{(\mathcal{J} \times s)^2} \cdot \underline{s} \qquad \text{erg/cm}^2\text{/sec.}$$

($\mathcal{J} \sim \omega \cdot$ Dipole moment)

whence $S \sim \omega^4$ (Dip. mom.)^2

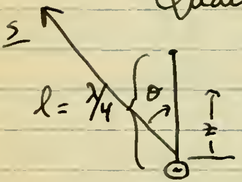
Per unit solid angle $d\Omega$:

$$\frac{c}{4\pi} d\Omega \; k^2 \overline{(\mathcal{J} \times s)^2} \qquad \text{erg/sec.}$$

or : $\frac{30}{4\pi} d\Omega \; k^2 \overline{(\mathcal{J}_a \times s)^2}$ { joule/sec
watt.

<u>Examples</u>

Quarter wave antenna



$$j_0 = j_z = \widehat{\text{const.}}^{\,a} \cos kz \qquad k = \frac{2\pi}{\lambda}$$

$$kl = \pi/2 \;;\; l = \lambda/4$$

then $\mathcal{J} = a \int_0^l dz \cos kz \, e^{-iks\cdot x'}$

(should really have different k - say k' for antenna)

$$= a \int_0^l \cos kz \, e^{-ikz\cos\theta} \, dz$$

$$= \frac{a}{2} \int_0^l \left[e^{ikz(1-\cos\theta)} + e^{-ikz(1+\cos\theta)} \right.$$

$$= \frac{a}{2ik} \; \frac{e^{ikz(1-\cos\theta)}}{1-\cos\theta} \Big|_0^l - \frac{e^{-ikz(1+\cos\theta)}}{1+\cos\theta} \Big|_0$$

$$= \frac{a}{2ik} \left[\frac{e^{ikl} \cdot e^{-ikl\cos\theta} - 1}{1-\cos\theta} - \frac{e^{-ikl} \, e^{-ikl\cos\theta} - 1}{1+\cos\theta} \right.$$

$$= \frac{a}{2ik} \left[\frac{(1+\cos\theta)(e^{ikl} e^{-ikl\cos\theta} - 1) - (1-\cos\theta)(e^{-ikl} \cdot e \ldots}{1-\cos^2\theta} \right.$$

$$= \frac{a}{2ik}\left[\frac{e^{-ikl\cos\theta}\left[e^{ikl}(1+\cos\theta)-e^{-ikl}(1-\cos\theta)\right]+2}{\sin^2\theta}\right]$$

$$= \frac{a}{2ik}\left[\frac{e^{-ikl\cos\theta}\left[2i\sin kl+2\cos\theta\cos kl\right]+2}{\sin^2\theta}\right]$$

but $kl = \pi/2$

$$\therefore\ J = \frac{a}{2ik}\left[\frac{e^{-\frac{i\pi}{2}\cos\theta}[2i]+2}{\sin^2\theta}\right]$$

$$= \frac{a}{ik\sin^2\theta}\left(i e^{-\frac{\pi}{2}i\cos\theta}-\cos\theta\right)$$

$$\underline{J}\times\underline{s} = J\sin\theta = \frac{a}{ik\sin\theta}\left(i e^{-\frac{\pi}{2}i\cos\theta}-\cos\theta\right)$$

$$(\ \ ) = i\cos\left(\tfrac{\pi}{2}\cos\theta\right)+\sin\left(\tfrac{\pi}{2}\cos\theta\right)-\cos\theta$$

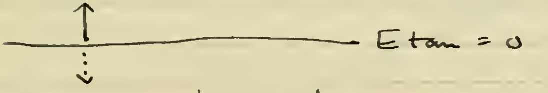
$$|\ \ |^2 = \cos^2\left(\tfrac{\pi}{2}\cos\theta\right)+\sin^2\left(\tfrac{\pi}{2}\cos\theta\right)-2\cos\theta\sin(\ \ )+\cos^2\theta$$

$$= 1+\cos^2\theta-2\cos\theta\sin\left(\tfrac{\pi}{2}\cos\theta\right)$$

$\cos\theta$ small: $|\ \ |^2 = 1-(\pi-1)\cos^2\theta$

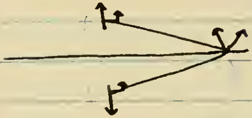
$$|\underline{J}\times\underline{s}|^2 = \frac{|\ \ |^2}{\sin^2\theta} = \frac{|\ \ |^2}{1-\cos^2\theta} = 1-(\pi-2)\cos^2\theta.$$

Earth is a conductor: ~~tangential~~ comp. of E vanishes;
∴ take image antenna:

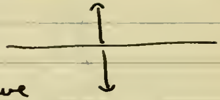


~~turns~~ out in above case that E at surface is normal to earth anyhow — but conductivity attenuates the wave.

Jan 19, 42 Antenna with earth: use mirror image:



charges symmetrical,
current not:

if on earth: 
get half-wave antenna -

then radiation becomes:

$kl = \pi/2$

$$\frac{e^{ikl(1-\cos\theta)} - e^{-ikl(1-\cos\theta)}}{1-\cos\theta} - \frac{e^{-ikl(1+\cos\theta)} - e^{ikl(1+\cos\theta)}}{1+\cos\theta}$$

$$= \frac{2i\sin[kl(1-\cos\theta)] + 2\cos\theta[\cos(kl(1-\cos\theta)] + 2i\sin(+ \ldots}{\sin^2\theta \qquad \mp 2\cos\theta[\cos \ldots}$$

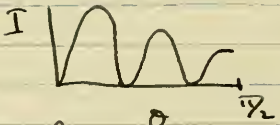
$$= \frac{2[\cos\theta\{\cos kl(-) + \cos kl(+)\} + i\sin\theta\{ \ldots}{\sin^2\theta}.$$

$$\sim \frac{2i\sin kl(1-\cos\theta)}{\sin^2\theta} \sim \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

If $kl = (n+\frac{1}{2})\pi$, get: $\frac{\cos((n+\frac{1}{2})\pi\cos\theta)}{\sin^2\theta}$.

$I(\theta) = 0$ for $(n+\frac{1}{2})\cos\theta = m+\frac{1}{2}$.

$\cos\theta = \frac{m+\frac{1}{2}}{n+\frac{1}{2}}$

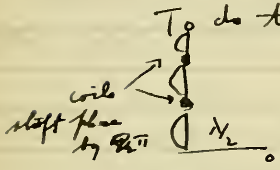


this not good: want large $I(\theta)$ for $\theta \sim 90°$.

$J(s) = \int j_0(r')\, e^{ik\, s\cdot r'}\, dr'$ if s is horizontal, r' is vertical then $s \cdot r' = 0$, $J = \int j_0 \ldots$

i.e. make current have same phase everywhere along antenna.
so 1/4 wave antenna is better than 3/4, etc —
But want long antenna to make J small for almost vertical $\underline{s}$; j_0 in phase with itself along antenna to make J large for horizontal $\underline{s}$.

To do this:



$i = \sin kz \qquad 0 < kz < \pi$

small contribution to $\int j_0 \, dz'$ from coil – neglect.

$i = \sin(kz - \lambda/2) \qquad \pi < kz < 2\pi$

Radiation: $$J = \sum_{n=-a}^{+a} \int_{-l/2}^{l/2} \cos kz' \, e^{-ik(nl+z')\cos\theta} \, dz'$$ (z' measured from center)

$kl = \pi$ $\quad$ ($2a+1$ sections)

$$J = \sum_{n=-a}^{a} e^{-iknl\cos\theta} \int_{-l/2}^{l/2} \cos kz' \, e^{-ikz'\cos\theta} \, dz'$$

$$= \sum_{n=-a}^{a} e^{-iknl\cos\theta} \cdot \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}.$$

$$= \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot e^{-i\pi\cos\theta a} \frac{e^{i(2a+1)\pi\cos\theta} - 1}{e^{\pi i\cos\theta} - 1}$$

$$= \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot \frac{\sin\{\pi \cos(a+\frac{1}{2})\theta\}}{\sin(\frac{\pi}{2}\cos\theta)}$$

$$= \frac{\cot(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \cdot \sin\{\pi\cos[(a+\tfrac{1}{2})\theta]\}$$

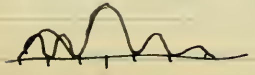
slowly varying ↑ $\qquad$ rapidly ↑ varying.

Max for $\theta = \pi/2$

1st Min. for $\cos\theta = \frac{1}{a+\frac{1}{2}}$.

2nd Max for $\cos\theta = \frac{3/2}{a+\frac{1}{2}}$

but much smaller.

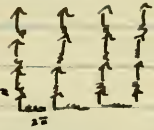


For larger no. of sections, more concentrated radiation (in $\theta = \pi/2$ direction)

This is still uniformly distributed in azimuth; if a directional antenna is desired, use two:

from in front, both are in phase; from the side, they are out of phase, & hence interfere destructively; this gives a fairly weak maximum.

add a third: ↑ ↑ ↑ get more pronounced maxima, & more minima; Make all these marconi antennas, & get only horizontal radiation in desired direction; in limit have a whole "diffraction pattern" of antennas:



But now come in large weight resistance & radiation damping.

Schelkunoff: radiation losses for single antenna I.R.E. (points out that radiated energy cannot be treated like joule heat, which was done previously)

Jan 21 '42 (Perkins)

Array of antennas:

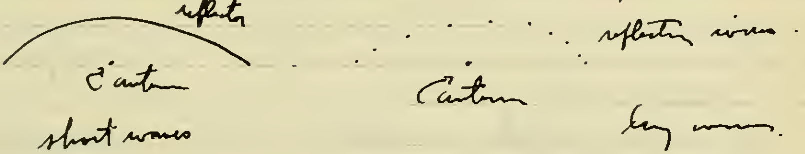
$$J = J_1 \sum_{m=1}^{M} \sum_{n=1}^{N} e^{-ik(s_y m l_y + s_z n l_z)}$$

where J_1 is radiation from single antenna.

$$J = J_1 \frac{\sin \frac{1}{2} M k l_y s_y}{\sin \frac{1}{2} k l_y s_y} \cdot \frac{\sin \frac{1}{2} N k l_z s_z}{\sin \frac{1}{2} k l_z s_z}$$

$l_y = l_z = \pi/k$ $\quad \therefore J = J_1 \dfrac{\sin \frac{M s_y}{2}\pi}{\sin \frac{\pi}{2} s_y} \cdot \dfrac{\sin \pi \frac{N s_z}{2}}{\sin \frac{\pi s_z}{2}}$.

Principal maxima at $\frac{1}{2} k l_y s_y = 0, \pi$
Want minim for $s_y = 1$ for directional effect (x-direction)
$\therefore \frac{1}{2} k l_y = \pi/2$. if $k l_y = 2\pi$ radiation in both x + y direction
reflectors also used for directional effects.

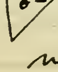


For long wave length, can shorten antenna by putting capacity on top

$\underline{J} = \int \underline{j} \cdot e^{i\kappa \underline{s}\cdot\underline{r}'} dr'$
expand exp: $e^{i\kappa \underline{s}\cdot\underline{r}'} = 1 + i\kappa(\underline{s}\cdot\underline{r}') - \frac{\kappa^2}{2}(\underline{s}\cdot\underline{r}')^2 \dots$

For long waves, small apparatus, can neglect retardation
$e^{i\kappa \underline{s}\cdot\underline{r}} \sim 1$; then $\underline{J} = \int \underline{j}_0 \, d\tau$ i.e. J is indep. of s.
~~then $E \sim J \times s$ ... $\sin\theta$ ... $S^2 \sim \sin^2\theta$~~

Poynting vector $\underline{S} \sim |J \times s|^2 \cdot \underline{s} \sim \sin^2\theta$

 For long λ, if $\int j_0 \, d\tau = 0$, radiation is much reduced; then must calculate the second term — "Quadrupole" radiation —

$J_q \sim J_d \,\kappa r$ $\qquad r$ - dimension of radiating system.

$S_q \sim S_d \,(\kappa r)^2$.

$\underline{J}_d = \int \underline{j}_0 \,\underline{s}\cdot\underline{r} = \underline{\underline{A}}\,\underline{s}$ $\qquad$ (A a dyadic).

$A_{ik} = \int j_{0i} r_k \left(\neq \int j_{0k} r_i \right)$ not symmetric

but can be separated into sym. & unsym. parts -

$A = B + C$. $\qquad B_{ik} = \frac{1}{2}\int (j_{0i} r_k + j_{0k} r_i)$

$C_{ik} = \frac{1}{2}$ " $-$ " .

$C = \frac{1}{2}\int j \times r$. magnetic moment of

magnetic dipole.

Two dipoles in opposite direction but displaced slightly give quadrupole radiation.

Similar to oscillations in a sphere:

E waves, $E_r \neq 0$ $H_r = 0$; H waves, $H_r \neq 0$, $E_r = 0$.

$$E_r = P_{lm}(\cos\theta)\, e^{im\varphi} f(r).$$

$$f'' + \frac{2}{r} f' + (\kappa^2 - \frac{l(l+1)}{r^2}) f = 0$$

No solution for $l = 0$;

otherwise $f \sim J_{l+\frac{1}{2}}(\kappa r)\, r^{\frac{1}{2}}$

for outgoing waves, use Hankel f'n:

$$f \sim H_{l+\frac{1}{2}}(\kappa r)\, r^{\frac{1}{2}}$$

$l = 1 \sim$ dipole rad.

$l = 2 \sim$ quadrupole rad.

$$e^{i\kappa \underline{s}\cdot\underline{r}} \sim \frac{1}{r^{\frac{1}{2}}} \sum (2l+1)\, J_{l+\frac{1}{2}}(\kappa r)\, P_l(\widehat{s,r})$$

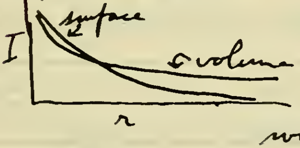
$$= \sim \sum_{l,m} J_{l+\frac{1}{2}}(\kappa r)\, P_{lm}(s)\, P_{lm}(r) \quad ?$$

$$\therefore J \sim \sum P_{lm}(s) \int j_0 \, d\tau \, \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}}(\kappa r)\, P_{lm}(r) \quad \sim (\kappa r)^l.$$

Jan 23 '42
Last day!

Propagation.

antenna + mirror image (with current reversed) gives E normal to earth's surface, H along earth ⊥ to propagation. If earth has finite conductivity (not perfect conductor) get absorption; in wave guide case we found absorption per km ~ $\frac{\omega^{3/2}}{\sqrt{\sigma}}$. One finds a "surface wave" which is guided along earth much like waves in wave guide. Turns out that $I \sim \frac{1}{r}$, ∴ $E, H \sim \frac{1}{\sqrt{r}}$, so surface waves decrease ~~in intensity~~ slower than the volume wave. If absorption, however, $I \sim \frac{1}{r} e^{-\beta r}$ for surface wave; volume wave, $I \sim 1/r^2$.



phases are different; may get interference between the two. Often a zone some distance from station, where practically no signal is heard.

Absorption increases with decreasing σ: on earth surface: sea water, Fresh water = Wet earth, Dry earth
little abs. — more abs. — most abs.

Spherical earth: Zenneck, Sommerfeld, Van der Pol. not of much significance.

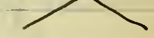
Reflection by ionized Heaviside layer —

measure time between receiving direct & reflected signal at pt. near transmitter: is ~ 1000 μsec, indicating height ~ 150 ~~300~~ km of layer.

Heaviside layer is due to ultra-violet in sun's radiation which ionizes N_2 + O_2, giving + ions + electrons. Recombination of ions takes a long time,

because (1) low pressure (2) small probability of capture of electron by ion; lifetime 10^{-8} sec for excited atom; 10^{-15} sec. is time it takes electron to traverse atom; so probability of radiation (i.e. capture of electron) is ~ 10^{-7} per traversal. Usually have: triple collision; energy instead of going into radiation, goes into kinetic energy of another molecule; but this also rare because of low pressure.

At night the lowest layer disappears because of recombination. Lifetime ~ a few hours in lower layers. (in upper layer, insignificant recombination in 12 hrs even)

Reflecting layer can also absorb, especially if it has not a well defined edge instead of .

In day, lower edge is more apt to be fuzzy in day time, hence better reception at night.

dielectric constant $\varepsilon = 1 - \frac{4\pi n e^2}{m\omega^2}$.

Scattering: in mountains reception bad because of scattering by mountains

Airplane detection; distinguish between $\lambda \gg a$ and $\lambda \ll a$ where a ~ size of scattering object. In latter case, geometrical optics applies; scattering cross section ~ a^2. In former case, would have again scattering ~ a^2 if "monopole" waves ($l=0$); but for $l=1$, which actually is the case, scattering ~ $\frac{a^6}{\lambda^4}$. [polarization/currents in object ~ E $\therefore M \sim E a^3$; and $E_{sc} \sim M k^2 \sim \frac{a^3}{\lambda^2}$ $\therefore$ intensity ~ $\frac{a^6}{\lambda^4}$]

$\lambda \sim a$ is desirable.

$$\underline{s}' = \underline{\underline{A}}\,\underline{s} =$$

$$= (\underline{p}\,\underline{q})\,s$$

$$= \underline{p}(\underline{q}\cdot\underline{s}) = (q_x s_x + q_y s_y + q_z s_z)\,p\,.$$